Higher Spin Holography

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Abstract

This dissertation splits into two distinct halves. The first half is devoted to the study of the holography of higher spin gauge theory in AdS_3. We present a conjecture that the holographic dual of W_N minimal model in a 't Hooft-like large N limit is an unusual "semi-local" higher spin gauge theory on $\mathrm{AdS}_3 \times \mathrm{S}^1$. At each point on the S^1 lives a copy of three-dimensional Vasiliev theory, that contains an infinite tower of higher spin gauge fields coupled to a single massive complex scalar propagating in AdS_3. The Vasiliev theories at different points on the S^1 are correlated only through the AdS_3 boundary conditions on the massive scalars. All but one single tower of higher spin symmetries are broken by the boundary conditions. This conjecture is checked by comparing tree-level two- and three-point functions, and also one-loop partition functions on both side of the duality. The second half focuses on the holography of higher spin gauge theory in AdS_4. We demonstrate that a supersymmetric and parity violating version of Vasiliev's higher spin gauge theory in AdS_4 admits boundary conditions that preserve $\mathcal{N} = 0, 1, 2, 3, 4$ or 6 supersymmetries. In particular, we argue that the Vasiliev theory with $U(M)$ Chan-Paton and $\mathcal{N} = 6$ boundary condition is holographically dual to the 2+1 dimensional $U(N)_k \times U(M)_{-k}$ ABJ theory in the limit of large N, k and finite M. In this system all bulk higher spin fields transform in the adjoint of the $U(M)$ gauge group, whose bulk t'Hooft coupling is $\frac{M}{N}$. Our picture suggests that the supersymmetric Vasiliev theory can be obtained as a limit of type IIA string theory in $\mathrm{AdS}_4 \times \mathbb{CP}^3$, and that the non-Abelian Vasiliev theory at strong bulk 't Hooft coupling smoothly turn into a string field theory. The fundamental string is a singlet bound state of Vasiliev's higher spin particles held together by $U(M)$ gauge interactions.

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Citations to Previously Published Work

Chapter 2 is essentially identical to

> "Higher Spin Gravity with Matter in AdS_3 and Its CFT Dual", C.-M. Chang and X. Yin, *JHEP* **1210** (2012) 024, [`arXiv:1106.2580`].

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Dedicated to my grandfather 張是欽,
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my mother 袁宇英,
my brother 張其昌,
and Justin.

Part I

AdS_3 higher spin holography

Chapter 1

Introduction and Summary

One of the greatest challenges in theoretical physics is formulating a quantum theory of gravity, a theory that would unify quantum mechanics and general relativity. Despite the fact that we live in a de Sitter space, quantum gravity in asymptotically anti-de Sitter space has instead undergone substantial development in the past decade due to the advance of AdS/CFT correspondence [1, 2, 3].

The AdS/CFT correspondence in principle gives a precise and non-perturbative formulation of quantum gravity in terms of large N gauge theories. In practice, our understanding of quantum gravity using AdS/CFT has been largely limited by difficulties in solving strongly coupled large N gauge theories. Thus, exactly solvable models of strongly coupled gauge theories with a semi-classical gravity dual are highly desirable. In two dimensions, there are lots of exactly solvable conformal field theories. Most of them do not have a large N limit that allows for a weakly coupled gravity dual. In [4], Gaberdiel and Gopakumar proposed that the coset models

$$\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}} \tag{1.1}$$

in the 't Hooft-like large N limit, where N, k are taken to infinity while fixing the 't Hooft coupling $\lambda = N/(k+N)$, are dual to *some* weakly coupled bulk theory . The central charge of the CFT is

$$c = (N-1)\left(1 - \frac{N(N+1)}{(N+k)(N+k+1)}\right) = N(1-\lambda^2) + \mathcal{O}(N^0). \tag{1.2}$$

The linear dependence on N is characteristic of a vector model. This coset model has a holomorphic spin-s current $W^{(s)}$ and an anti-holomorphic spin-s current $\overline{W}^{(s)}$ for each spin $s = 2, 3, 4, \cdots, N$. The Fourier modes of $W^{(s)}$ generate the W_N algebra, which is a higher spin generalization of the Virasoro algebra. The coset models (1.1) are usually refereed to as the W_N minimal model. In the large N limit, the W_N algebra turns into the $W_\infty[\lambda]$ algebra that contains generators with arbitrary spins. In W_N minimal model, the W_N primary operators, the primaries with respect to the W_N algebra, are labeled by two representatons (Λ_+, Λ_-), where $\Lambda_\pm$ are the highest weight representations of $SU(N)_k$ and $SU(N)_{k+1}$, respectively.[1] For fixed representations Λ_+, Λ_- at sufficiently large N,[2] the fusion coefficients for the primary operators in the W_N minimal model is simply given by the product of the fusion coefficients in the $SU(N)_k$ and $SU(N)_{k+1}$ WZW models, i.e.

$$\mathcal{N}^{W_N}_{(\Lambda^1_+,\Lambda^1_-)(\Lambda^2_+,\Lambda^2_-)}{}^{(\Lambda^3_+,\Lambda^3_-)} = \mathcal{N}^{(k)}_{\Lambda^1_+\Lambda^2_+}{}^{\Lambda^3_+}\mathcal{N}^{(k+1)}_{\Lambda^1_-\Lambda^2_-}{}^{\Lambda^3_-}, \tag{1.3}$$

where $\mathcal{N}^{(k)}_{\Lambda^1\Lambda^2}{}^{\Lambda^3}$ is the fusion coefficient of $SU(N)_k$ WZW model.

The gravity dual of W_N minimal model at large N is a higher spin gauge theory, which contains a tower of gauge fields of spins $s = 2, 3, 4, \cdots, \infty$ that are dual to the higher spin

[1] A prior, the primary should also depend on the highest weight representation Λ_0 of $SU(N)_1$. However, Λ_0 can be determining by requiring $\Lambda_+ + \Lambda_0 - \Lambda_-$ being inside the root lattice of $SU(N)$.

[2] Namely representations that are found in the tensor product of finitely many fundamental or anti-fundamental representations of $SU(N)$, at large N.

currents $W^{(s)}$ and $\overline{W}^{(s)}$. The pure higher spin gauge theory on AdS_3 can be described by the Chern-Simons action with $hs(\lambda) \times hs(\lambda)$ gauge algebra. The higher spin algebra $hs(\lambda)$ is an infinite dimensional Lie algebra, and by a Brown-Henneaux type computation, it was shown, in [5, 6, 7], that $W_\infty[\lambda]$ is the the asymptotic symmetry algebra of higher-spin gravity based on the algebra $hs(\lambda)$. It also follows from this computation that the bulk coupling constant is proportional to inverse the square root of the central charge, i.e.

$$g_{bulk} \sim \frac{1}{\sqrt{c}} \sim \frac{1}{\sqrt{N}}. \tag{1.4}$$

The primary operators in the W_N minimal model, constructed from the diagonal modular invariant, do not carry spin. They should be dual to scalar elementary particles and their bound states with zero angular momentum, that become unbound in the infinite N (zero bulk coupling) limit. In particular, the primary operator $\phi_1 = (\square, 0)$ is dual to a scalar field with left and right conformal dimension equal to

$$h_{(\square,0)} = \frac{1}{2}(1+\lambda) \tag{1.5}$$

in the large N limit. The primary $\bar{\phi}_1 = (\bar{\square}, 0)$ has the same dimension in the large N limit, and is dual to the anti-particle of $(\square, 0)$. The primary operators $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, 0)$ and $(\square\square, 0)$ have conformal weights

$$h_{(\begin{smallmatrix}\square\\\square\end{smallmatrix},0)} = 1+\lambda, \quad h_{(\square\square,0)} = 2+\lambda \tag{1.6}$$

in the large N limit. Note that $h_{(\begin{smallmatrix}\square\\\square\end{smallmatrix},0)}$ and $h_{(\square\square,0)}$ are twice the dimension of $(\square, 0)$ plus a non-negative integer. This allows for the identification of $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, 0)$ and $(\square\square, 0)$ as two-particle states of ϕ_1's. In general, the primary operators of the form $(\Lambda, 0)$ are dual to the multi-particle states of $B(\Lambda)$ ϕ_1's, where $B(\Lambda)$ is the number of boxes of the Young tableaux of the representation Λ (here we assume that $B(\Lambda)$ does not scale with N). The W_N minimal model in the large N limit has a symmetry that exchanges Λ_+ with Λ_-, while flipping the sign of λ. Hence, the primary $\tilde{\phi}_1 = (0, \square)$ is dual to a scalar elementary particle, with dimension

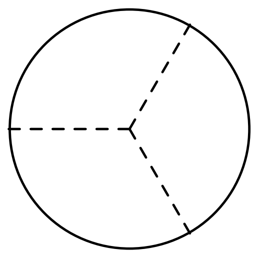
$$h_{(0,\square)} = \frac{1}{2}(1-\lambda), \tag{1.7}$$

and the primaries $(0, \Lambda)$ are dual to the multi-particle states of $\tilde{\phi}_1$. The fusion coefficients (1.3) implies that the primaries of the form $(\Lambda, 0)$ (or $(0, \Lambda)$) are closed under the OPE, as long as Λ is small compared to N. They form a closed subsector of the W_N minimal model in the large N limit. Either one of these two subsectors has a consistent set of n-point functions on the sphere, in the sense that they factorize through only operators within the same subsector. In Chapter 2, we proposed a bulk dual for each of the subsectors. The classical bulk theory is described by Vasiliev's system in three dimensions [8, 9, 10], which is a higher spin gauge theory of gauge fields of spin $s = 2, 3, \cdots, \infty$ based on the higher spin algebra $hs(\lambda)$, coupled to a complex massive scalar field of mass squared $m^2 = -(1-\lambda^2)$. This conjecture has also been checked by matching the three-point function $\left\langle \phi_1 \bar{\phi}_1 W^{(s)} \right\rangle$ computed on both side of the correspondence in Chapter 2 and [10, 11].

To go beyond these two subsectors, in Chapter 3, we study the bulk dual of the class of primary operators (Λ_+, Λ_-) for $\Lambda_\pm$ being one- or two-box representations. In this class of primaries, we identify a number of single-trace operators, which are dual to single-particles states in the bulk. They are summarized as follows,

$$\begin{aligned}
&\phi_1 = (\square, 0), \quad \tilde{\phi}_1 = (0, \square), \quad \omega_1 = (\square, \square), \\
&\phi_2 = \frac{1}{\sqrt{2}}\left[(\square\!\square, \square) - (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square)\right], \quad \tilde{\phi}_2 = \frac{1}{\sqrt{2}}\left[(\square, \square\!\square) - (\square, \begin{smallmatrix}\square\\\square\end{smallmatrix})\right], \\
&\omega_2 = \frac{1}{\sqrt{2}}\left[(\square\!\square, \square\!\square) - (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\end{smallmatrix})\right].
\end{aligned} \tag{1.8}$$

$\phi_1, \tilde{\phi}_1, \phi_2, \tilde{\phi}_2$ have spin zero and dimension of order 1 in the large N limit. They are dual to massive scalars in the bulk theory. ω_1, ω_2 have spin zero and dimension of order $1/N$. They are dual to massless scalars in the bulk. By analyzing exact results of three-point functions, in particular, we demonstrate that the three-point function of three single-trace operators in (1.8) is of order $1/\sqrt{N}$ in the large N limit. This agrees with our expectation from the bulk Witten's diagram of three single elementary particles in a weakly coupled theory,



$$\sim \frac{1}{\sqrt{N}} \sim g_{bulk}.$$

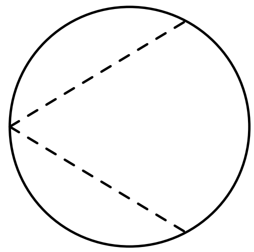
All the other primary operators are identified as multi-trace operators, which are dual to multi-particle states in the bulk. They are summarized in the following table.

$\Lambda_+ \backslash \Lambda_-$	0	$\square$	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
0	$\mathbf{1}$	$\tilde{\phi}_1$	$L_{\tilde{\phi}_1}$	$\tilde{\phi}_1^2$
$\square$	ϕ_1	ω_1	$\frac{1}{\sqrt{2}}(\tilde{\phi}_1\omega_1 + \tilde{\phi}_2)$	$\frac{1}{\sqrt{2}}(\tilde{\phi}_1\omega_1 - \tilde{\phi}_2)$
$\square\square$	L_{ϕ_1}	$\frac{1}{\sqrt{2}}(\phi_1\omega_1+\phi_2)$	$\frac{1}{2}(\omega_1^2 + \sqrt{2}\omega_2)$	$\frac{1}{\sqrt{2}}(L_{\omega_1}-\frac{1}{\sqrt{2}}(\phi_1\tilde{\phi}_2-\phi_2\tilde{\phi}_1))$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	ϕ_1^2	$\frac{1}{\sqrt{2}}(\phi_1\omega_1-\phi_2)$	$\frac{1}{\sqrt{2}}(L_{\omega_1}+\frac{1}{\sqrt{2}}(\phi_1\tilde{\phi}_2-\phi_2\tilde{\phi}_1))$	$\frac{1}{2}(\omega_1^2 - \sqrt{2}\omega_2)$

The operator $L_{\mathcal{O}}$ is defined as

$$L_{\mathcal{O}} = \frac{1}{2\sqrt{2}h_{\mathcal{O}}} \left(\mathcal{O}\partial\bar{\partial}\mathcal{O} - \partial\mathcal{O}\bar{\partial}\mathcal{O}\right), \tag{1.9}$$

which is dual to an excited state of a two-particle state in the bulk. Consider two single-trace operators, for example ϕ_1 and ω_1 in (1.8), the single-particle states dual to ϕ_1 and ω_1 can form a bound state, which is dual to a double-trace operator $\frac{1}{\sqrt{2}}[(\square\square,\square) + (\begin{smallmatrix}\square\\\square\end{smallmatrix},\square)]$. By analyzing the exact three-point functions, we demonstrate in Section 3.4 that the three point function of ϕ_1, ω_1, and $\frac{1}{\sqrt{2}}[(\square\square,\square) + (\begin{smallmatrix}\square\\\square\end{smallmatrix},\square)]$ is of order 1 in the large N limit. This agrees with the bulk Witten's diagram of two elementary particles with their bound state,

 $\sim\ 1.$

Our identification of single-trace operators versus multi-trace oprators is subject to a peculiar relation [12, 50]:

$$\frac{1}{2h_{\omega_1}}\partial\bar{\partial}\omega_1 = \phi_1\tilde{\phi}_1, \quad h_{\omega_1} = \frac{\lambda}{2N}, \tag{1.10}$$

which, although naively seems to be in conflict with large N factorization, has a very natural bulk interpretation that will be discussed later.

In Section 4.2, we carry on the identification of single-trace operators for the class of primaries that includes also the operators with Λ_+ or Λ_- being three-box representations. We find three more single-trace operators ϕ_3, $\tilde{\phi}_3$ and ω_3,

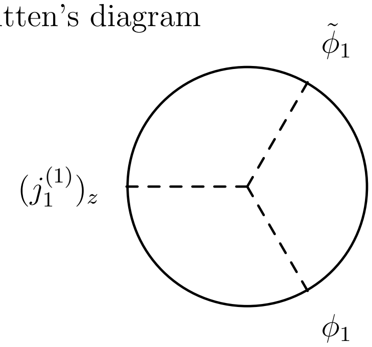
$$\begin{aligned}
\phi_3 &= \frac{1}{\sqrt{6}}\left[\sqrt{2}(\yng(3),\yng(2)) - (\yng(2,1),\yng(2)) - (\yng(2,1),\yng(1,1)) + \sqrt{2}(\yng(1,1,1),\yng(1,1))\right], \\
\tilde{\phi}_3 &= \frac{1}{\sqrt{6}}\left[\sqrt{2}(\yng(2),\yng(3)) - (\yng(2),\yng(2,1)) - (\yng(1,1),\yng(2,1)) + \sqrt{2}(\yng(1,1),\yng(1,1,1))\right], \\
\omega_3 &= \frac{1}{\sqrt{3}}\left[(\yng(3),\yng(3)) - (\yng(2,1),\yng(2,1)) + (\yng(1,1,1),\yng(1,1,1))\right],
\end{aligned} \tag{1.11}$$

and all the other primary operators are identified as multi-trace operators. The large N factorization has also been check for this larger class of primaries. In the large N limit, $\phi_n, \tilde{\phi}_n$ have the same value of dimension and higher spin charges as $\phi_1, \tilde{\phi}_1$, and the dimension and higher spin charges of ω_n are n times bigger than the corresponding values for ω_1. It is very tempting to conjecture that the single-trace operators of finite dimension in the large N limit fall into the three classes ϕ_n, $\tilde{\phi}_n$ and ω_n for n being positive integers. ϕ_n is a linear combination of primaries (Λ_+, Λ_-) with $(n, n-1)$ boxes, $\tilde{\phi}_n$ is a linear combination of primaries with $(n-1, n)$ boxes, and ω_n is a linear combination of primaries (Λ, Λ) with

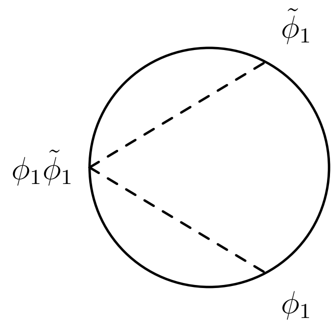
Λ being n-box representations, which has dimension $\sim n/N$ in the large N limit. However, this is not the full story; there are more single-trace operators. The key observation is that (1.10) can be interpreted as a *current non-conservation equation*,

$$\bar{\partial}(j_1^{(1)})_z = \frac{\lambda}{\sqrt{N}}\phi_1\tilde{\phi}_1, \tag{1.12}$$

where $j_1^{(1)} = (j_1^{(1)})_z dz + (j_1^{(1)})_{\bar{z}} d\bar{z} = (\partial\omega_1 dz + \bar{\partial}\omega_1 d\bar{z})/\sqrt{2h_{\omega_1}}$ is the level-1 descendent of ω_1 with normalized two-point function. In the infinite N limit, the right hand side of (1.12) vanishes, and $(j_1^{(1)})_z$ becomes a *primary* spin-1 current. We refer these kind of operators as *large N primary operators*, the operators that effectively become primary fields in the infinite N limit. We propose that the bulk dual of $(j_1^{(1)})_z$ is a $U(1)$ Chern-Simons gauge field A_μ coupled to two scalar fields Φ and $\widetilde{\Phi}$, which are dual to ϕ_1 and $\tilde{\phi}_1$, respectively. Φ and $\widetilde{\Phi}$ have the same mass but satisfy different boundary condition (fall-off behavior near the AdS boundary), which however is incompatible with the $U(1)$ gauge transformation generated by A_μ. As a result, the $U(1)$ gauge symmetry, though is conserved in the bulk classically, is broken by $1/N$ effects induced by the scalar boundary conditions; hence, is hidden from the boundary CFT point of view. This entire picture is checked in Section 4.4 by an explicit bulk computation of the Witten's diagram



which after taking the $\bar{\partial}$-derivative is proportional to the factorized Witten's diagram,



This computation essentially reproduces the current non-conservation equation (1.12). In Section 4.3 and Section 4.5, we demonstrate that the level-1 descendants of ω_2, ω_3 and also a level-2 descendant of ω_1 satisfy similar current non-conservation equations as (1.12). We propose that the bulk dual of them are Chern-Simons spin-1 gauge fields and also a spin-2 gauge field in the bulk.

The amount of evidences are enough for us to present a consistent conjecture in Section 4.6, that for each ω_n there exist a tower of large N primaries $j_n^{(s)}$, which are conserved spin $s \geq 1$ currents in the infinite N limit. The complete spectrum of single-trace operator of W_N minimal model is then given by a tower of spin-0 W_N primaries $\phi_n, \tilde{\phi}_n, \omega_n$ and a tower of spin-s large N primaries $j_n^{(s)}$, all of which are complex. In Section 4.7, we provide a highly nontrivial check on this spectrum of single-trace operators, by matching the the torus partition of W_N minimal in the infinite N limit with the bulk one-loop partition function given by this spectrum.

The approximately conserved spin-s currents $j_n^{(s)}$ are dual to gauge fields in AdS$_3$ of various spins, which generate hidden higher spin gauge symmetries in the bulk. The massive scalars dual to $\phi_n, \tilde{\phi}_n$ are charged under the hidden higher spin gauge symmetries. In Section 4.8, we determine the gauge generators associated with the hidden symmetry currents, which are incompatible with the boundary conditions on the massive scalars and leads to the breaking of symmetry.

Our conjecture on the large N spectrum, combined with the identification of the gauge generators acting on the matter scalars, leads to a dramatically new picture of the holographic dual of the W_N minimal model. We propose that the dual higher spin gauge theories is a "semi-local"[3] theory living on $\mathrm{AdS}_3 \times \mathrm{S}^1$. This is not an ordinary four-dimensional field theory, however. At each point of the S^1, there is a tower of higher spin gauge fields in AdS_3, coupled to a single complex massive scalar field, of the type described by Vasiliev's system in three dimensions. The different Vasiliev theories at different points on the S^1 appear to be decoupled at the level of bulk equations of motion. Rather, they interact only through the boundary condition which mixes scalar fields living at different points on the circle S^1. Essentially, while all the scalars classically have the same mass in AdS_3, the boundary condition assigns one scaling dimension $2h_\phi$ on right-moving modes of the scalar on the circle, and the complementary scaling dimension $2h_{\tilde{\phi}} = 2 - 2h_\phi$ on left-moving modes of the scalar on the circle.

While our proposal for the holographic dual is rather unconventional due to the large degeneracy in the bulk fields, it seems to be unavoidable due to peculiarities in the structure of large N factorization in W_N minimal model. We believe that it is characteristic of gauged vector models on non-simply connected spaces [14, 15]. Presumably, what we see here is the field theory of the tensionless limit of a more conventional string theory in AdS_3, dual to quiver-like generations of the W_N minimal model, and the S^1 should come from a topological sector of the string theory in this limit.

[3]The terminology comes from analogy with the holographic theory of semi-local quantum liquids [13].

Chapter 2

Higher Spin Gravity with Matter in AdS_3 and Its CFT Dual

2.1 Introduction

The AdS/CFT correspondence [1, 2, 3] has given us a tremendous amount of insight in quantum gravity through its duality with large N gauge theories. Progress does not come easily, however. The regime in which the bulk theory reduces to semi-classical gravity is typically dual to a gauge theory in the strong 't Hooft coupling regime, and is difficult to solve. In the opposite limit, where the gauge theory is weakly coupled, the bulk theory is typically in a very stringy regime, involving strings in AdS whose radius is very small in string units (though large in Planck units, as long as N is large). With a few exceptions, such as the purely NS-NS background of AdS_3 [16], in which case the dual CFT is singular [17, 18], generally the bulk string theory involves Ramond-Ramond fluxes; even the free string spectrum is difficult to solve, and the full string field theory appears to be out of reach at the moment.

A particularly simple class of conjectured AdS/CFT dualities [19, 20, 4] avoids these difficulties. These involve boundary CFTs whose numbers of degrees of freedom scales like N rather than N^2. In the $\mathrm{AdS}_4/\mathrm{CFT}_3$ conjecture of [19], the boundary theory is given by the critical $O(N)$ vector model. Such a duality can be extended to Chern-Simons-matter theories with vector matter representations [21]. In the $\mathrm{AdS}_3/\mathrm{CFT}_2$ conjecture of [4], the boundary theory is the W_N minimal model, which can be realized as the coset model

$$\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}}. \tag{2.1}$$

In these examples, the CFT is either exactly solvable or has a simple $1/N$ expansion that can be computed straightforwardly order by order. The dual bulk theories, however, are higher spin extensions of gravity, involving an infinite tower[1] of higher spin gauge fields. In the case of [4], additional massive scalar matter fields are coupled to the higher spin gauge fields. It is likely that these higher spin gauge theories are UV complete (at least perturbatively) theories that contain gravity, due to the large number of gauge symmetries, and are interesting toy models for quantum gravity. However, they do not reduce to semi-classical gravity in any limit. Note that the higher spin symmetry can be broken by AdS boundary conditions [19, 23], but this breaking is controlled by the coupling constant of the theory and is in some sense rather mild.

The goal of the current paper is to understand the conjectured duality of [4] at the interacting level, in particular, to the second order in perturbation theory. In fact, a careful

[1]While a *pure* higher spin gauge theory in AdS_3 involving spins up to N can be formulated in terms of $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ Chern-Simons theory, it is not known how to couple this theory to scalar matter fields. The construction of [22] requires an infinite set of gauge fields of spins $s = 2, 3, \cdots, \infty$. This is the system conjectured to be dual to the W_N minimal model in [4]. While the dynamical mechanism that renders the set of spins finite in the interacting theory has not yet been understood, this seeming mismatch is not visible at any given order in perturbation theory.

examination of the spectrum of the linearized Vasiliev system leads us to propose a modification of the conjecture of [4]. A key insight of [4] is that, in the large N limit of the coset model (2.1), $\lambda = N/(N+k)$ plays the role of the 't Hooft coupling, and the basic primaries labelled by representations $(\Box;0)$ and $(0;\Box)$ (as well as the conjugate representations) have finite scaling dimensions Δ_+ and Δ_- in the 't Hooft limit, and are conjectured to be dual to massive scalars in the bulk. We will consider a version of Vasiliev's system that involve a gauge field of spin s for $s = 2, 3, \cdots, \infty$, coupled to two real massive scalar fields. We propose that it is dual to a *subsector* of the W_N minimal model, generated by the W_N currents together with two basic primary operators of dimension Δ_+, labelled by $(\Box;0)$ and $(\overline{\Box};0)$, or two basic primaries of dimension Δ_- labelled by $(0;\Box)$ and $(0;\overline{\Box})$, depending on the boundary condition imposed on the bulk scalar. We will refer to these two subsectors as the Δ_+ subsector and the Δ_- subsector, respectively. Each subsector has closed OPEs, and hence consistent n-point functions on the sphere, in the sense that they only factorize through operators within in the same subsector. This identification is natural by comparing the bulk fields and boundary operators, and also avoids the puzzle with "light states" in the 't Hooft limit of the coset model.[2] However, it suggests that the bulk Vasiliev system is non-perturbatively incomplete, though makes sense to all order in perturbation theory. It may be possible to enlarge Vasiliev's system to obtain a higher spin-matter theory that is dual to the full W_N minimal model, but such a bulk theory would be subject to the strange feature of having a large number of light states. We will not address this possibility in the current paper. There is, on the other hand, a minimal truncation of Vasiliev system, where

[2]The "light states" are the primaries labelled by a pair of identical representations, $(R;R)$, whose dimension scales like $1/N$ in the large N limit. While the contribution of such states to the partition function is argued in [4] to decouple in the strict infinite N limit, they show up in OPEs of basic primaries when $1/N$ corrections are taken into account.

one keeps only the even spin fields and one out of the two real massive scalars. We conjecture that this system is dual to the orthogonal group version of the W_N minimal model.[3]

The main nontrivial check of our proposal is a comparison of the tree level three-point functions involving two scalars and one higher spin field in the bulk, and the 't Hooft limit of the corresponding three point function in the dual CFT. In order to carry out such a computation, we first solve for the boundary to bulk propagators of Vasiliev's master fields, and then expand the nonlinear equations of motion to second order in perturbation theory and compute the three point function. We encounter subtleties with gauge ambiguity and boundary condition on the higher spin fields, and will find explicit formulae for the gauge field propagators obeying the boundary condition of [5]. While one may expect that, in principle, such three point functions are determined by symmetries and Ward identities, the implementation of the latter is not so trivial on the CFT side. For instance, we do not know a simple way to carry out the $1/N$ expansion of the coset model, and must calculate correlators exactly at finite N first, and then take the 't Hooft limit. For various quantities of interest in the CFT, analytic formulae for general spins are often difficult to obtain, and instead one computes case by case for the first few spins. The results have a nontrivial dependence on the 't Hooft coupling λ, which is mapped to a deformation parameter ν in the bulk theory. The case in which the bulk theory is the simplest, namely the $\nu = 0$ "undeformed" theory, is mapped to $\lambda = 1/2$. In this paper, most of our computation is performed within the $\nu = 0$ theory, and is compared to the $\lambda = 1/2$ case of the W_N minimal model. In Appendix 2.C we give some formulae useful for the deformed bulk theory with nonzero ν, though the analogous computation of correlators in the deformed theory is left

[3] The 't Hooft limit of this class of CFTs are recently studied in [24].

to future work.

More precisely, we compute correlators of the form $\langle \overline{\mathcal{O}}\mathcal{O}J^{(s)}\rangle$ at tree level in the $\nu = 0$ undeformed bulk theory. These three-point functions are fixed by conformal symmetry up to the overall coefficient; the latter is computed unambiguously as a function of the spin s. The result is then compared to the three point functions in the W_N minimal model, in the large N limit, at 't Hooft coupling $\lambda = 1/2$. We test the conjectured duality using the explicit expression for the spin 3 current in the coset construction, and found perfect agreement.

We begin with a brief review of the three-dimensional Vasiliev's system in Section 2.2. In Section 2.3 we describe the linearized spectrum of the bulk theory, as well as propagators and boundary conditions, while leaving technical details to Appendix 2.A. Some useful formulae for the deformed bulk theory (i.e. with nonzero ν) are given in Appendices 2.C. In Section 2.4, we work to second order in perturbation theory and compute the three point functions of interest. The details of these derivations are given in Appendix 2.B. Our proposal of the dualities and a test on the three point functions are presented in Section 2.5. We conclude in Section 2.6.

2.2 A brief review of Vasiliev's system in AdS$_3$

Throughout this paper, we will consider the Vasiliev system in AdS$_3$, which consists of one higher spin gauge field for each spin $s = 2, 3, 4, \cdots$, coupled to a pair of real massive scalar fields. We will often work explicitly with the Poincaré coordinates of AdS$_3$, with $x^\mu = (z, x^i)$, $i = 1, 2$, and the metric $ds^2 = \frac{1}{z^2}(dz^2 + dx^i dx^i)$. Following Vasiliev, we introduce the auxiliary bosonic twistor variables y_α, z_α, where $\alpha = 1, 2$ is a spinorial index, as well as the Grassmannian variables ψ_i, $i = 1, 2$, which obey $\{\psi_i, \psi_j\} = 2\delta_{ij}$.[4] The master fields are: W a 1-form in the spacetime parameterized by x^μ, S a 1-form in the auxiliary z^α-space, and B a scalar field. All of them are functions of $x^\mu, y_\alpha, z_\alpha$, as well as ψ_i,[5]

$$\begin{aligned} &W = W_\mu(x|y, z, \psi_i)dx^\mu, \\ &S = S_\alpha(x|y, z, \psi_i)dz^\alpha, \\ &B = B(x|y, z, \psi_i). \end{aligned} \tag{2.2}$$

These fields are subject to a large set of gauge symmetries. The infinitesimal gauge transformation is parameterized by a function $\epsilon(x|y, z, \psi)$,

$$\begin{aligned} &\delta W = d_x\epsilon + [W, \epsilon]_*, \\ &\delta S = d_z\epsilon + [S, \epsilon]_*, \\ &\delta B = [B, \epsilon]_*. \end{aligned} \tag{2.3}$$

One further imposes a truncation so that W, B are even functions of (y, z) whereas S_α is odd in (y, z) (so that the 1-form S is even under $(y, z, dz) \mapsto (-y, -z, -dz)$). The gauge parameter ϵ is then restricted to be an even function of (y, z) as well. One introduces a star-product $*$ on functions of (y, z), defined by

$$f(y, z) * g(y, z) = \int d^2ud^2ve^{uv}f(y + u, z + u)g(y + v, z - v). \tag{2.4}$$

Here and throughout this paper, the spinors are contracted as $uv = u^\alpha v_\alpha = -v^\alpha u_\alpha = -vu$ and $u\sigma v = u^\alpha \sigma_\alpha{}^\beta v_\beta$ for a matrix σ. The integration measure d^2ud^2v above is normalized

[4]Note that while the equations of motion treats ψ_1 and ψ_2 on equal footing, the choice of vacuum will not. The ψ_i's can be thought of as purely a bookkeeping device.

[5]In Vasiliev's original papers, the master fields depend on the additional Grassmannian variables k, ρ. This will be discussed in Appendix 2.C. We will refer it as the "extended Vasiliev system", the Vasiliev system we present here is obtained by making a projection $(1 + k)/2$ on all fields, and effectively eliminating k, ρ.

such that $f * 1 = f$. The Grassmannian variables ψ_i commute with y_α, z_α and do not participate in the $*$ product. Under the star-product, the auxiliary variables y_α generate the three dimensional higher spin algebra $hs(1,1)$ [25][6], which is an associative algebra, whose general element can be represented by a even analytic function of in y_α. In particular, $hs(1,1)$ has a subalgebra $sl(2)$ whose generator can be written as $T_{\alpha\beta} = y_{(\alpha} * y_{\beta)}$. An inner product on this algebra is defined as $(A, B) = A(y) * B(y)\big|_{y=0}$.

We define an involution ι on the star algebra as follows: $\iota(y^\alpha) = iy^\alpha$, $\iota(z^\alpha) = -iz^\alpha$, $\iota(dz^\alpha) = -idz^\alpha$, and the action of ι reverses the order of all products (including the multiplication of ψ_i's); in particular, $\iota(\psi_1\psi_2) = \psi_2\psi_1 = -\psi_1\psi_2$. The master fields W, S, B are then subject to the reality condition[7]

$$\iota(W)^* = -W, \quad \iota(S)^* = -S, \quad \text{and} \quad \iota(B)^* = B, \tag{2.5}$$

where the superscript $*$ stands for taking the complex conjugate on the component fields while leaving the auxiliary variables $y^\alpha, z^\alpha, \psi_i$ untouched.

Vasiliev's equations of motion are now written as

$$\begin{aligned}
&d_x W + W * W = 0, \\
&d_x S + d_z W + \{W, S\}_* = 0, \\
&d_z S + S * S = B * K dz^2, \\
&d_x B + [W, B]_* = 0, \\
&d_z B + [S, B]_* = 0.
\end{aligned} \tag{2.6}$$

[6]We will also consider $hs(\lambda)$ the one parameter deformation of $hs(1,1)$ in Appendix 2.C.

[7]Such a reality condition is necessary because, as we will see later, the physical components of the B master field are of the form $\psi_2 C_{even} + \psi_2\psi_1 C_{odd}$ where C_{even} is a real scalar and C_{odd} is a purely imaginary scalar field.

Here d_x and d_z denote the exterior derivative in spacetime coordinates x^μ and the auxiliary variables z^α respectively. $K \equiv e^{zy}$ is known as the Kleinian. It has the properties

$$K * K = 1, \qquad K * f(y,z) = K f(z,y), \qquad f(y,z) * K = K f(-z,-y). \tag{2.7}$$

A few comments on (2.6) are in order. The third equation in (2.6) can be thought of as the definition of the scalar master field B. The fourth equation is equivalent to a Bianchi identity for the field strength of the connection $\mathcal{A} = W + S$, which follows from the second and third equation. The last equation, however, is an independent equation for B.[8]

Note that the equations of motion (2.6) are preserved under the involution ι, if one sends (W, S, B) to $(-W, -S, B)$ at the same time. In particular, Vasiliev's system can be further truncated down to what we refer to as the "minimal Vasiliev's system". The latter is defined by projecting the master fields onto the ι-invariant components, namely

$$\iota(W) = -W, \quad \iota(S) = -S, \quad \text{and} \quad \iota(B) = B. \tag{2.8}$$

We will see later that the minimal Vasiliev's system contains only the even spin gauge fields and a single matter scalar. Though, in most of this paper, we will be considering the untruncated Vasiliev's system, where gauge spins of all spins greater than or equal to 2 are included.

The equations (2.6) are formulated in a background independent manner. To formulate the perturbation theory, one begins by choosing a vacuum solution, and identifies the physical propagating degrees of freedom by linearizing the equations around the vacuum solution. One may then proceed to higher orders in perturbation theory and study interactions in this

[8]This is different from the four-dimensional version of Vasiliev's system, which involves a similar set of equations.

background. It turns out that the system (2.6) admits a 1-parameter family of distinct AdS_3 vacua, labeled by a real parameter ν. In fact, the parameter ν appears in a non-dynamical, auxiliary component of B, and thus the 1-parameter family of AdS_3 vacua are not connected by physical deformations, but should rather be thought of as different theories in AdS_3. In this paper, we will focus on the simplest, "undeformed" theory, corresponding to the $\nu = 0$ vacuum. The deformed vacua/theories ($\nu \neq 0$) are discussed in Appendix 2.C. The perturbation theory, and in particular the study of three point functions, of the *deformed* theory is left to future work.

The undeformed AdS_3 vacuum solution is given by

$$B = 0, \qquad S = 0, \qquad W = W_0 \equiv w_0(x|y) + \psi_1 e_0(x|y), \tag{2.9}$$

where W_0 is a flat connection satisfying $d_x W_0 + W_0 * W_0 = 0$. With $W_0(x|y, \psi_1)$ chosen to be a quadratic function of y, the flatness condition is classically equivalent to the Chern-Simons formulation of Einstein's equation with negative cosmological constant in three dimensions. In other words, the equations of motion is obeyed if the 1-forms e_0, w_0 are chosen as the dreibein and spin connection for AdS_3, contracted with y^α in spinorial notation. In Poincaré coordinates $x^\mu = (z, x^i)$, they can be written as

$$w_0(x|y) \equiv w_0^{\alpha\beta}(x) y_\alpha y_\beta = -\frac{y\sigma^{\mu z} y}{8z} dx^\mu, \qquad e_0(x|y) \equiv e_0^{\alpha\beta}(x) y_\alpha y_\beta = -\frac{y\sigma^\mu y}{8z} dx^\mu. \tag{2.10}$$

Our convention for e_0 is such that

$$(e_0^\mu)_{\alpha\beta}(e_{0\mu})^{\gamma\delta} = -\frac{1}{64}(\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma), \qquad (e_0^\mu)_{\alpha\beta}(e_{0\nu})^{\alpha\beta} = -\frac{1}{32}\delta_\nu^\mu. \tag{2.11}$$

Expanding around this vacuum solution, we will write $W = W_0 + \widehat{W}$, and the equations of motion in its perturbative form as

$$
\begin{aligned}
&D_0\widehat{W} = -\widehat{W} * \widehat{W},\\
&D_0 S + d_z\widehat{W} = -\{\widehat{W}, S\}_*,\\
&d_z S - B * K dz^2 = -S * S,\\
&d_z B = -[S, B]_*,\\
&D_0 B = -[\widehat{W}, B]_*,
\end{aligned}
\tag{2.12}
$$

where we have defined $D_0 \equiv d_x + [W_0, \cdot]_*$. By choosing a z_α-dependent gauge function, one can always go to a gauge in which $S|_{z_\alpha=0} = 0$. The physical degrees of freedom are entirely contained in the z_α-independent part of the master fields, whereas the z_α-dependence are determined via the equations of motion. It is then useful to decompose W, B as

$$
\begin{aligned}
W(x|y,z,\psi) &= W_0 + \Omega(x|y,\psi) + W'(x|y,z,\psi)\\
B(x|y,z,\psi) &= C(x|y,\psi) + B'(x|y,z,\psi)
\end{aligned}
\tag{2.13}
$$

where Ω and C are the restriction of $\widehat{W}$ and B to $z_\alpha = 0$, respectively, while W' and B' obey $W'\big|_{z_\alpha=0} = B'\big|_{z_\alpha=0} = 0$. We will see that Ω and C contain the higher spin gauge fields and two real scalar fields, whereas W' and B' are auxiliary fields. At the linearized level, the equations (2.12) reduce to

$$D_0\Omega^{(1)} = -\{W_0, W'^{(1)}\}_*|_{z=0}, \tag{2.14}$$

$$d_z W'^{(1)} = -D_0 S^{(1)}, \tag{2.15}$$

$$d_z S^{(1)} = C^{(1)} * K dz^2, \tag{2.16}$$

$$B'^{(1)} = 0, \tag{2.17}$$

$$D_0 C^{(1)} = 0, \tag{2.18}$$

where the superscript (n) labels the order of the component of the respective field in the perturbative expansion. These equations will be analyzed in detail in the next section as well as in Appendix 2.A. We will then proceed to the quadratic order and study the cubic coupling and three point functions in Section 2.4.

Let us note that the system of equations (2.6) and the AdS$_3$ vacuum (2.9) are invariant under a global $U(1)$ symmetry,

$$W \to e^{i\theta\psi_1} W e^{-i\theta\psi_1}, \quad S \to e^{i\theta\psi_1} S e^{-i\theta\psi_1}, \quad B \to e^{i\theta\psi_1} B e^{-i\theta\psi_1}. \tag{2.19}$$

This $U(1)$ rotates the phase of the complex scalar matter field, while leaving the higher spin fields invariant. Note that (2.19) preserves the reality condition (2.5). While it is a symmetry of the classical theory, and is expected to be a perturbative symmetry of the quantum theory, it should be broken non-perturbatively (or alternatively, become gauged), as anticipated in any quantum theory of gravity [26, 27]. In the proposed dual CFT, the $U(1)$ rotates the basic primaries $(\Box; 0)$ and $(\overline{\Box}; 0)$ with opposite phases. As far as correlators of a fixed number of basic primaries are concerned, in the large N limit, this $U(1)$ is effectively a symmetry of the theory, since any correlation function that violates the $U(1)$ vanishes by the fusion rule. This $U(1)$ is obviously broken when N basic primaries are inserted, as the tensor product of N fundamental representations of $SU(N)$ contains a singlet.

2.3 Propagators and two point functions

2.3.1 The physical fields and propagators

In this subsection we will describe the physical degrees of freedom in the linearized master fields, as well as their propagators. The details of the derivations starting from Vasiliev's equation are given in Appendix 2.A.

The scalar matter field

The linearized scalar master field $C^{(1)}(x|y,\psi)$ can be decomposed as

$$C^{(1)}(x|y,\psi_i) = C^{(1)}_{aux}(x|y,\psi_1) + \psi_2 C^{(1)}_{mat}(x|y,\psi_1). \tag{2.20}$$

$C^{(1)}_{aux}$ is purely auxiliary; the only solution to its equation of motion is a constant, which parameterizes a family of AdS_3 vacua. We will set $C^{(1)}_{aux} = 0$ for now. $C^{(1)}_{mat}$ can be expanded in y as

$$C^{(1)}_{mat} = \sum C^{(1),n}_{mat}(x|y,\psi_1) = \sum C^{(1),n}_{mat\ \alpha_1\cdots\alpha_n}(x|\psi_1) y^{\alpha_1}\cdots y^{\alpha_n}. \tag{2.21}$$

It follows from $D_0(\psi_2 C^{(1)}_{mat}) = 0$ that the bottom component $C^{(1),0}_{mat}(x|\psi_1)$ obeys the usual Klein-Gordon equation for a massive scalar field in AdS_3,

$$\left(\nabla^\mu \partial_\mu - m^2\right) C^{(1),0}_{mat}(x|\psi_1) = 0, \quad m^2 = -\frac{3}{4}. \tag{2.22}$$

Expanding further in ψ_1, $C^{(1),0}_{mat}(x|\psi_1) = C_{even}(x) + \psi_1 C_{odd}(x)$ contain a pair of real scalars of mass squared $m^2 = -\frac{3}{4}$ in AdS units. Due to the reality condition (2.5), C_{even} is real whereas C_{odd} is a purely imaginary scalar field. They can be paired up to a complex massive scalar as $C_{even} + C_{odd}$, with $C_{even} - C_{odd}$ its complex conjugate. Under the global $U(1)$ symmetry (2.19), $C_{even} \pm C_{odd}$ transform as

$$C_{even} \pm C_{odd} \to e^{\pm i\theta}\left(C_{even} \pm C_{odd}\right). \tag{2.23}$$

In the dual boundary CFT, this complex scalar corresponds to a complex scalar operator of dimension Δ_+ or Δ_-, depending on the choice of boundary condition. Here

$$\Delta_\pm = 1 \pm \frac{1}{2} = \frac{3}{2} \text{ or } \frac{1}{2}. \tag{2.24}$$

The higher components $C_{mat}^{(1),n}$ are expressed in terms of derivatives of $C_{mat}^{(1),0}$ through the equation of motion.

In the ν-deformed vacua, $C_{mat}^{(1)}$ still describes a pair of real massive scalar fields, with mass squared $m^2 = -\frac{3}{4} + \frac{\nu(\nu\pm 2)}{4}$, where the $\pm$ sign depends on a choice of projection. This is discussed in Appendix 2.C.

The boundary-to-bulk propagator for the scalar is $C^{mat,0} = K(\vec{x},z)^{\Delta}$ for $\Delta = 3/2$ or $\Delta = 1/2$, where $K(\vec{x},z) \equiv \frac{z}{\vec{x}^2+z^2}$, $\vec{x} = (x^1, x^2)$. It is convenient to introduce another auxiliary variable $\widetilde{\psi}_1$, satisfying $\widetilde{\psi}_1^2 = 1$, to label the two different boundary conditions, so that $\Delta = 1 + \widetilde{\psi}_1/2$. With the δ-function source on C_{even} component:

$$C_{mat}^{(1)}(\vec{x}, z \to 0|y, \psi_1) = 2\pi \widetilde{\psi}_1 z^{1-\frac{\widetilde{\psi}_1}{2}} \delta^2(x) \tag{2.25}$$

turned on on the boundary, the boundary-to-bulk propagator for the master field $C_{mat}^{(1)}(x|y,\psi_1)$ is then given by

$$C_{mat}^{(1)}(x|y,\psi_1) = \left(1 + \psi_1 \frac{1+\widetilde{\psi}_1}{2} y\Sigma y\right) e^{\frac{\psi_1}{2} y\Sigma y} K^{1+\frac{\widetilde{\psi}_1}{2}}, \tag{2.26}$$

where $\Sigma \equiv \sigma^z - \frac{2z}{x^2}\sigma^\mu x^\mu$. We can also turn on the source on C_{odd} component:

$$C_{mat}^{(1)}(\vec{x}, z \to 0|y, \psi_1) = 2\pi \psi_1 \widetilde{\psi}_1 z^{1-\frac{\widetilde{\psi}_1}{2}} \delta^2(x) \tag{2.27}$$

on the boundary. The boundary-to-bulk propagator will be just (2.26) times ψ_1.

Under the action of the involution ι, C_{even} is invariant whereas C_{odd} changes sign. Hence only C_{even} survives the minimal truncation (2.8). Thus, the "minimal Vasiliev system" contains only a single real scalar scalar, which is dual to a real scalar operator in the boundary CFT. Note that in writing the boundary-to-bulk propagator (2.26), we have chosen to turn on a source for C_{even} only, and the result is invariant under the projection by ι.

The higher spin fields

The higher spin gauge fields, as well as some auxiliary fields, are contained in $\Omega(x|y,\psi)$, which may be decomposed in the form

$$\Omega^{(1)}(x|y,\psi_i) = \Omega^{hs}(x|y,\psi_1) + \psi_2\Omega^{sc}(x|y,\psi_1). \tag{2.28}$$

As the notations suggest, Ω^{hs} contain the higher spin gauge fields in AdS$_3$, while Ω^{sc} are in fact auxiliary fields determined by the scalar matter fields. The linearized equations take the form

$$D_0\Omega^{hs} = 0, \quad \tilde{D}_0\Omega^{sc} = -\psi_2\{W_0, \psi_2 W^{mat}\}_*|_{z=0}. \tag{2.29}$$

where we have defined

$$\tilde{D}_0 \equiv d_x + [w_0, \cdot]_* - \psi_1\{e_0, \cdot\}_*. \tag{2.30}$$

It is demonstrated in Appendix 2.A.2 that up to gauge transformations, Ω^{sc} have no propagating degrees of freedom and are determined entirely in terms of C_{mat}. Ω^{hs}, on the other hand, obeys the (linearized) Chern-Simons equation with higher spin algebra $\mathrm{hs}(1,1)\oplus\mathrm{hs}(1,1)$. They are related to the metric-like higher spin fields, which are usually written in terms of traceless symmetric tensors, in the following way.

First, expand $\Omega^{hs}_{\alpha\beta} \equiv \Omega^{hs}_\mu (e_0^\mu)_{\alpha\beta}$ in y as

$$\Omega^{hs}_{\alpha\beta}(x|y,\psi_1) = \sum \Omega^{hs,(n)}_{\alpha\beta}(x|y,\psi_1) = \sum \Omega^{hs,n}_{\alpha\beta|\alpha_1\cdots\alpha_n}(x|\psi_1) y^{\alpha_1}\cdots y^{\alpha_n}, \tag{2.31}$$

and then express the components in terms of symmetric traceless tensors (in spinorial notation) as

$$\Omega^{hs,(n)}_{\alpha\beta|\alpha_1\cdots\alpha_n}(x|\psi_1) = \chi^{n,+}_{\alpha\beta\alpha_1\cdots\alpha_n} + \epsilon_{(\alpha_1(\underline{\alpha}}\chi^{n,0}_{\underline{\beta})\alpha_2\cdots\alpha_n)} + \epsilon_{(\underline{\alpha}(\alpha_1}\epsilon_{\underline{\beta})\alpha_2}\chi^{n,-}_{\alpha_3\cdots\alpha_n)}, \tag{2.32}$$

or equivalently,

$$\Omega^{hs,(n)}_{\alpha\beta}(x|y,\psi_1) = \frac{1}{(n+2)(n+1)}\partial_\alpha\partial_\beta\chi^+_n(x|y,\psi_1) + \frac{1}{n}y_{(\alpha}\partial_{\beta)}\chi^0_n(x|y,\psi_1) + y_\alpha y_\beta\chi^-_n(x|y,\psi_1). \tag{2.33}$$

Here $\chi^+_n(x|y,\psi_1)$ is defined as $\chi^{n,+}_{\alpha_1\cdots\alpha_{n+2}}$ contracted with y^α's, and similarly for $\chi^0_n(x|y,\psi_1)$ and $\chi^-_n(x|y,\psi_1)$. Next, we expand in ψ_1, and write

$$\chi^{\pm/0}_n = \chi^{n,\pm/0}_{even} + \psi_1\chi^{n,\pm/0}_{odd}. \tag{2.34}$$

It turns out that χ_{even} are determined in terms of (derivatives of) χ_{odd} through the equation of motion. Furthermore, $\chi^{n,0}_{odd}$ can be gauged away entirely. The residual gauge symmetry on $\chi^{n,\pm}_{odd}(y)$ takes the form

$$\begin{aligned} \delta\chi^{n,+}_{odd}(y) &= -\nabla^+\lambda^n_{odd}(y), \\ \delta\chi^{n,-}_{odd}(y) &= -\frac{1}{n(n+1)}\nabla^-\lambda^n_{odd}(y), \end{aligned} \tag{2.35}$$

where $\lambda^n_{odd}(y)$ is related to the gauge parameter ϵ by $\epsilon = \psi_1\lambda^n_{odd}$. $\nabla^\pm$ are defined here as

$$\nabla^+ \equiv (ye^\mu_0 y)\nabla_\mu, \quad \nabla^- \equiv (\partial_y e^\mu_0\partial_y)\nabla_\mu, \tag{2.36}$$

where ∇_μ acts on a tensor $(\cdots)_{\alpha_1\alpha_2\cdots}$ as the spin-covariant derivative. Under the ι-action, only the even spin fields are invariant. Hence, the "minimal" Vasiliev's system only contains higher spin gauge fields with even spins, and its dual boundary CFT contains only even spin currents.

In the metric-like formulation, the spin-s gauge field is described by a rank s double traceless symmetric tensor $\Phi_{\mu_1\cdots\mu_s}$. It may be decomposed into irreducible representations of the Lorentz group as

$$\Phi_{\mu_1\cdots\mu_s} = \xi_{\mu_1\cdots\mu_s} + g_{(\mu_1\mu_2}\chi_{\mu_3\cdots\mu_s)}, \tag{2.37}$$

where ξ and χ are traceless symmetric tensors of rank s and $s-2$, respectively. With the identification

$$\chi_{odd}^{2s-2,+} = \xi^{(s)}, \qquad \chi_{odd}^{2s-2,-} = -\frac{2s-3}{32(s-1)}\chi^{(s)}, \tag{2.38}$$

where $\xi^{(s)}$ is defined as $\xi_{\mu_1\cdots\mu_s}$ contracted with $(e_0^{\mu})_{\alpha\beta}y^{\alpha}y^{\beta}$, and similarly for $\chi^{(s)}$, the Chern-Simons form of the equations of motion can be shown to be equivalent to the Fronsdal form of the equation on Φ,

$$\begin{aligned}(\Box - m^2)\Phi_{\mu_1\cdots\mu_s} - s\nabla_{(\underline{\mu_1}}\nabla^{\mu}\Phi_{\mu\underline{\mu_2\cdots\mu_s})} + \frac{1}{2}s(s-1)\nabla_{(\underline{\mu}_1}\nabla_{\underline{\mu}_2}\Phi^{\mu}{}_{\mu\underline{\mu_3\cdots\mu_s})}\\ - s(s-1)g_{(\underline{\mu_1\mu_2}}\Phi^{\mu}{}_{\mu\underline{\mu_3\cdots\mu_s})} = 0,\end{aligned} \tag{2.39}$$

which is invariant under the gauge transformation:

$$\delta\Phi_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\eta_{\mu_2\cdots\mu_s)}, \tag{2.40}$$

where $\eta_{\mu_2\cdots\mu_s}$ is a symmetric traceless gauge parameter. The gauge transformation (2.40) is also equivalent to (2.35) under the identification (2.38).

In three dimensions, the higher spin gauge fields do not have bulk propagating degrees of freedom. In AdS$_3$, just as in the more familiar case of gravitons ($s=2$), there are boundary excitations of the higher spin fields, corresponding to field configurations that cannot be gauged away by gauge transformations that vanish on the boundary of the AdS spacetime. A careful analysis of the gauge conditions is necessary in order to talk about boundary-to-bulk propagators and bulk-to-bulk propagators. We will first consider Metsaev's modified de Donder gauge [28], which is convenient for solving higher spin propagators in AdS in general dimensions. We will see, however, that the propagators found in this gauge violates (the higher spin generalization of) Brown-Henneaux boundary condition, and are not directly applicable to the computation of boundary correlators. Nonetheless, this gauge should be useful in doing loop computations in the bulk. We will then proceed to find the appropriate boundary-to-bulk propagators that obey Brown-Henneaux boundary condition, which allows for computations of boundary correlators.

2.3.2 Propagators in modified de Donder gauge

The modified de Donder gauge was introduced by Metsaev in [28]. This gauge has the advantage that the equations of motion for different components of free higher spin gauge fields decouple, and hence the solutions can be obtained easily. The implementation of the gauge condition, on the other hand, is a bit complicated. It can be described as follows. Start with the double traceless symmetric $\Phi^s_{\mu_1\cdots\mu_s}$ which obeys the Fronsdal equation in AdS$_3$. Write $\Phi^s_{A_1\cdots A_s} = \Phi^s_{\mu_1\cdots\mu_s} e^{\mu_1}_{A_1} \cdots e^{\mu_s}_{A_s}$ where A_i are local Lorentz frame indices. Define a generating function/field

$$\Phi^s(x|Y) = \Phi^s_{A_1\cdots A_s} Y^{A_1} \cdots Y^{A_s}, \tag{2.41}$$

where $Y^A = (Y^z, Y^1, Y^2)$ are auxiliary vector variables (analogous to the twistor variables y^α introduced previously). One then performs a linear transformation on $\Phi^s(x|Y)$,

$$\phi(x|Y) = z^{-\frac{1}{2}} \mathcal{N} \Pi^{\phi\Phi} \Phi^s(x|Y), \tag{2.42}$$

where z is the Poincaré radial coordinate, $\mathcal{N}$ is an operator that acts as a separate normalization factor on each component of $\Phi(x|Y)$ of given degree in Y^z and $\vec{Y} = (Y^1, Y^2)$, and $\Pi^{\phi\Phi}$ involves derivatives on Y^z and $\vec{Y}$. See Appendix 2.A.3 for the definition of these operators. The resulting generating field $\phi(x|Y)$ is double traceless with respect to the directions parallel to the boundary, namely

$$\left(\frac{\partial^2}{\partial \vec{Y}^2}\right)^2 \phi(x|Y) = 0. \tag{2.43}$$

The modified de Donder gauge is defined by a gauge condition of the form

$$\overline{C}\phi(x|Y) = 0, \tag{2.44}$$

where $\overline{C}$ is an operator involving up to two derivatives on $\vec{Y}$ and one spacetime derivative. The key point is that, in this case, the Fronsdal equation for Φ^s is re-expressed in terms of equations on $\phi(x|Y)$ as

$$\left[\Box + \partial_z^2 - \frac{(r-\frac{1}{2})(r-\frac{3}{2})}{z^2}\right]\phi_r(x|\vec{Y}) = 0, \tag{2.45}$$

where $\phi_r(x|\vec{Y})$ are the components of $\phi(x|Y)$ expanded in Y^z,

$$\phi(x|Y) = \sum_{r=0}^{s}(Y^z)^{s-r}\phi_r(x|\vec{Y}). \tag{2.46}$$

The equation of motion is then straightforwardly solved in momentum space. Note that the gauge condition (2.44) relates the different components $\phi_r(x|\vec{Y})$. After solving $\phi(x|Y)$, one can translate it back into $\Phi^s(x|Y)$, and further into the frame-like fields $\chi^{(s),\pm}_{odd}$. The result for the boundary-to-bulk propagator of $\chi^{(s),\pm}_{odd}$ due to a chiral spin-s current $J^{(s)}_{++\cdots+}$ source inserted at $\vec{x}=0$ is given in momentum space explicitly by (up to the overall normalization factor)

$$\begin{aligned}
\chi^{(s),+}_{odd}(\vec{p},z|y) =& \sum_{r=0}^{s} i^r \binom{s}{r} p^{r-1}(p^+)^{s-r}(y^1)^{s+r}(y^2)^{s-r} z K_{r-1}(z|\vec{p}|),\\
\chi^{(s),-}_{odd}(\vec{p},z|y) =& \frac{s}{2(2s-1)}\sum_{r=0}^{s} i^r \binom{s-2}{r} p^{r-1}(p^+)^{s-r}(y^1)^{s+r-2}(y^2)^{s-r-2} z K_{r-1}(z|\vec{p}|).
\end{aligned} \tag{2.47}$$

The details of the derivation is given in Appendix 2.A.3. These propagators, however, do not obey the higher spin analog [5, 6] of Brown-Henneaux boundary condition [29], which should be imposed in order for the dual CFT to have the appropriate higher spin symmetry. In fact, we know that any solution to the linearized higher spin equations in AdS$_3$ must be a pure gauge in the bulk. The key to finding the appropriate boundary-to-bulk propagator is then to find the appropriate gauge transformation near the boundary. In the next subsection, we will see that such a gauge transformation takes a rather simple form. The bulk-to-bulk propagators in the modified de Donder gauge may still prove useful for loop computations in the bulk, which we hope to revisit in the future.

2.3.3 The asymptotic boundary condition

Let us begin with the spin 2 case, and consider the Brown-Henneaux boundary condition [29] on metric fluctuations. In the Y-algebra language, a spin 2 tensor field sourced by a positively polarized stress-energy tensor insertion on the boundary, at $\vec{x} = 0$, that obeys Brown-Henneaux boundary condition is given by

$$\Phi^2(x|Y) \sim \delta^2(\vec{x})(Y^+)^2 + (\text{subleading contact terms}) + \frac{z^2}{(x^-)^4}(Y^-)^2. \tag{2.48}$$

On the RHS we only indicated the leading order terms in the $z \to 0$ limit; their coefficients are not specified. The boundary-to-bulk propagators in the modified de Donder gauge, derived in the previous subsection, does not obey this boundary condition. It suffices to examine the spin 2 case. In position space, the graviton boundary to bulk propagator in the modified de Donder gauge (for a positively polarized source) is

$$\Phi^2(Y) = \frac{2i}{\pi} Y^z Y^+ \frac{x^+ z}{(x^2+z^2)^2} - \frac{i}{\pi}(Y^+)^2 \frac{z^2}{(x^2+z^2)^2} + \frac{i}{\pi} Y^+ Y^- \frac{(x^+)^2}{(x^2+z^2)^2}. \tag{2.49}$$

In the limit $z \to 0$, it goes like

$$\Phi^2(Y) \sim \delta^2(x)(Y^+)^2 + (\text{subleading contact terms}) + \frac{Y^- Y^+}{(x^-)^2}, \tag{2.50}$$

which clearly violates the boundary behavior of (2.48).

Similarly, the higher spin gauge fields are subject to the an analog of the Brown-Henneaux boundary conditions [5, 6]. For general spin s, the boundary condition is such that the boundary-to-bulk propagator for a positive polarized spin-s source is

$$\Phi^s(x|Y) \sim z^{2-s}\delta^2(\vec{x})(Y^+)^s + (\text{subleading contact terms}) + \frac{(Y^-)^s z^s}{(x^-)^{2s}}, \tag{2.51}$$

where the coefficient are again not specified. Let us examine this boundary condition (2.51) in more detail. In three dimension, similarly to gravitons, the higher spin gauge fields do not have any propagating degrees of freedom in the bulk. In other words, any solution to the equation of motion can be (locally) written in a pure gauge form, $\Phi^s(x|Y) = Y^A D^A \eta^s(x|Y)$. However, the gauge parameter $\eta^s(x|Y)$ may have nonzero higher spin charge, the latter is given by a boundary integral, and the higher spin gauge field $\Phi^s(x|Y)$ would not be gauge equivalence to zero. As proposed in [5], the boundary behavior of the gauge parameter $\eta^s(x|Y)$ can be fixed by demanding the gauge field $\Phi^s(x|Y)$ obeys the boundary conditions (2.51). With some effort, we find the appropriate gauge parameter $\eta^s(x|Y)$ near the boundary:

$$\begin{aligned}\eta^s(x|Y) = \sum_{u=0}^{s-1}\sum_{r=1}^{2s-2u-1}\sum_{v=0}^{u}\frac{(-1)^{r+u}}{(2u)!}\binom{u}{v}\left(\prod_{j=0}^{2u-1}(r+j)\right)\left(\prod_{j=1}^{u}\frac{2j-1}{2s-2j-1}\right)\\ \times (Y^3)^{2v+r-1}(Y^-)^{u-v}(Y^+)^{s-r-v-u}\frac{z^{2u+r-s}}{(x^-)^{2u+r}} + \mathcal{O}(z^{s+1}),\end{aligned} \tag{2.52}$$

and the corresponding gauge field

$$\begin{aligned}\Phi^s(x|Y) &= Y^A D^A \eta^s(x|Y)\\ &= 2\pi z^{2-s}\delta^2(x)(Y^+)^s + (\text{subleading contact terms})\\ &\quad + (-1)^s(2s-1)\frac{(Y^-)^s z^s}{(x^-)^{2s}} + \mathcal{O}(z^{s+1}).\end{aligned} \tag{2.53}$$

Notice that the leading analytic term on the RHS of (2.53) is proportional to the two point function of the boundary higher spin currents. Since the gauge parameter is a traceless tensor, i.e. $\partial_Y^2 \eta_s(Y) = 0$, we can substitute $Y^A = e^A_{\alpha\beta} y^\alpha y^\beta$ in (2.52) and obtain, modulo an overall normalization coefficient, the gauge parameter in the (spinorial) y-algebra language (see (2.35)):

$$\lambda^s(y) = -4 \sum_{r=1}^{2s-1} (y^1)^{2s-r-1} (y^2)^{r-1} \frac{z^{r-s}}{(x^-)^r} + \mathcal{O}(z^{s+1}). \tag{2.54}$$

For later use, we also compute the boundary-to-bulk propagators for the generating function of frame-like fields, $\chi_{odd}^{(s),\pm/0}$ and $\chi_{even}^{(s),\pm/0}$ using (2.143) and (2.138), and compute $\Omega_{11}^{hs,(s)}$ and $\Omega_{22}^{hs,(s)}$ using (2.134). They are

$$\begin{aligned}
\chi_{odd}^{(s),+} &= 2\pi (y^1)^{2s} z^{2-s} \delta^2(x) + (\text{subleading contact terms}) + \frac{(2s-1)(y^2)^{2s} z^s}{(x^-)^{2s}} + \mathcal{O}(z^{s+1}),\\
\chi_{odd}^{(s),0} &= 0,\\
\chi_{odd}^{(s),-} &= (\text{contact terms of the order } z^{4-2s} \text{ and higher}) + \mathcal{O}(z^{s+1}),
\end{aligned} \tag{2.55}$$

and

$$\begin{aligned}
\chi_{even}^{(s),+} &= -2\pi (y^1)^{2s} z^{2-s} \delta^2(x) + (\text{subleading contact terms}) - \frac{(2s-1)(y^2)^{2s} z^s}{(x^-)^{2s}} + \mathcal{O}(z^{s+1}),\\
\chi_{even}^{(s),0} &= (\text{contact terms of the order } z^{3-2s} \text{ and higher}) + \mathcal{O}(z^{s+1}),\\
\chi_{even}^{(s),-} &= (\text{contact terms of the order } z^{4-2s} \text{ and higher}) + \mathcal{O}(z^{s+1}),
\end{aligned} \tag{2.56}$$

as well as

$$\begin{aligned}
\Omega_{11}^{hs,(s)}(y) &= -2(1-\psi_1)\pi (y^1)^{2s-2} z^{2-s} \delta^2(x) + (\text{subleading contact terms}) + \mathcal{O}(z^{s+1}),\\
\Omega_{22}^{hs,(s)}(y) &= (\text{contact terms of the order } z^{4-s} \text{ and higher}) - (1-\psi_1) \frac{(2s-1)(y^2)^{2s-2} z^s}{(x^-)^{2s}} + \mathcal{O}(z^{s+1}).
\end{aligned} \tag{2.57}$$

Notice that the leading contact term in $\Omega_{11}^{hs,(s)}$ is proportional to $(1-\psi_1)$; in other words, we have imposed the Dirichlet boundary condition on the component $(1-\psi_1)\Omega_{11}^{hs,(s)}$. Similarly, for the negative polarized higher spin gauge field, we impose the Dirichlet boundary condition on the component $(1+\psi_1)\Omega_{22}^{hs,(s)}$.

2.3.4 Higher spin two point function

With these formulae at hand, we can now compute the two point function of the higher spin currents on the boundary. The linearized higher spin equation $D_0\Omega^{hs} = 0$ can be obtained from the quadratic part of a Chern-Simons type action:

$$S_{hs} = -\int d\psi_1 \int \left(\Omega^{hs}, d\Omega^{hs} + 2W_0 * \Omega^{hs}\right). \tag{2.58}$$

We decompose the higher spin gauge field as

$$\Omega^{hs} = \Omega_z^{hs} dz + \Omega_+^{hs} dx^+ + \Omega_-^{hs} dx^-. \tag{2.59}$$

Modulo the equation of motion, the variation of the action (2.58) is

$$\delta S_{hs} = -\int d\psi_1 \int dx^+ dx^- \frac{1}{z^2} \left[\left(\Omega_+^{hs}, \delta\Omega_-^{hs}\right) - \left(\Omega_-^{hs}, \delta\Omega_+^{hs}\right)\right], \tag{2.60}$$

which, however, is non-vanishing under the boundary condition (2.57). To cancel it, we add a boundary term to the action:

$$S_{hs,b} = -\int d\psi_1 \int dx^+ dx^- \frac{1}{z^2} \psi_1 \left(\Omega_+^{hs}, \Omega_-^{hs}\right), \tag{2.61}$$

whose variation is

$$\delta S_{hs,b} = -\int d\psi_1 \int dx^+ dx^- \frac{1}{z^2} \psi_1 \left[\left(\Omega_+^{hs}, \delta\Omega_-^{hs}\right) + \left(\Omega_-^{hs}, \delta\Omega_+^{hs}\right)\right]. \tag{2.62}$$

Hence, the variation of the total action $S_{hs} + S_{hs,b}$ is

$$\delta S_{hs} + \delta S_{hs,b} = -\int d\psi_1 \int dx^+ dx^- \frac{1}{z^2} \left[(1+\psi_1)\left(\Omega_+^{hs}, \delta\Omega_-^{hs}\right) - (1-\psi_1)\left(\Omega_-^{hs}, \delta\Omega_+^{hs}\right)\right]. \tag{2.63}$$

which indeed vanishes under the boundary condition (2.57), or equivalently the Dirichlet boundary condition on the components $(1-\psi_1)\Omega_+^{hs}$ and $(1+\psi_1)\Omega_-^{hs}$.

Since the bulk action (2.58) vanishes on-shell, the only contribution to the two-point function comes from the boundary term (2.61). Evaluating the boundary integral (2.61) using the higher spin boundary-to-bulk propagators, we obtain the two point function of higher spin currents:

$$\begin{aligned}\langle J_s(x_1)J_s(x_2)\rangle &= \int d^2x \frac{1}{z^2} 4\pi(\partial_{y^2})^{2s-2} z^{2-s}\delta^2(x-x_1)\frac{(2s-1)(y^2)^{2s-2}z^s}{(x^- - x_2^-)^{2s}} \\ &= 4\pi\frac{(2s-1)!}{(x_{12}^-)^{2s}}.\end{aligned} \tag{2.64}$$

This is indeed the structure expected from conformal invariance.

2.4 Three point functions

2.4.1 The second order equation for the scalars

To extract the cubic couplings in the bulk Lagrangian, or the three point correlation function of boundary operators, we need to express the master fields in terms of the physical fields and expand the equations of motion to quadratic order. For the purpose of studying three point functions involving the scalars, it suffices to work with the equations for the master field B, to the second order. They are

$$\begin{aligned} d_z B^{(2)} &= -[S^{(1)}, B^{(1)}]_*, \\ D_0 B^{(2)} &= -[W^{(1)}, B^{(1)}]_*. \end{aligned} \tag{2.65}$$

Decomposing $W^{(1)}, B^{(1)}, B^{(2)}$ as in (2.13), and restricting the second equation at $z=0$, we obtain

$$
\begin{aligned}
d_z B'^{(2)} &= -[S^{(1)}, \psi_2 C^{(1)}_{mat}]_*, \\
D_0 C^{(2)} &= -[W_0, B'^{(2)}]_*\big|_{z=0} - [W'^{(1)}, \psi_2 C^{(1)}_{mat}]_*\big|_{z=0} \\
&\qquad - [\Omega^{hs}, \psi_2 C^{(1)}_{mat}]_* - [\psi_2 \Omega^{sc}, \psi_2 C^{(1)}_{mat}]_*.
\end{aligned}
\tag{2.66}
$$

We remind the reader that $C^{(1)} = C^{(1)}_{aux} + \psi_2 C^{(1)}_{mat}$ and $\Omega^{(1)} = \Omega^{hs} + \psi_2 \Omega^{sc}$, and we have set $C^{(1)}_{aux} = 0$. The $S^{(1)}$ and $W'^{(1)}$ are linear in ψ_2, and the first equation implies $B'^{(2)}$ is independent of ψ_2. Decomposing $C^{(2)}$ in a similar way as $C^{(2)}(x|y,\psi) = C^{(2)}_{aux}(x|y,\psi_1) + \psi_2 C^{(2)}_{mat}(x|y,\psi_1)$, we obtain the second order equation for the scalars:

$$
D_0 \psi_2 C^{(2)}_{mat} = -[\Omega^{hs}, \psi_2 C^{(1)}_{mat}]_*,
\tag{2.67}
$$

or more explicitly

$$
D_0 \psi_2 C^{(2)}_{mat} = -\psi_2 [\Omega^{even}, C^{(1)}_{mat}]_* + \psi_2 \psi_1 \{\Omega^{odd}, C^{(1)}_{mat}\}_*,
\tag{2.68}
$$

where Ω^{even} and Ω^{odd} are the components in the decomposition $\Omega^{hs} = \Omega^{even} + \psi_1 \Omega^{odd}$.

We further decompose $C^{(2)}_{mat}$ as $C^{(2)}_{mat}(y) = \sum_{n=0}^{\infty} C^{(2),n}_{mat\ \alpha_1\cdots\alpha_n} y^{\alpha_1}\cdots y^{\alpha_n}$, and specialize (2.68) to the case $n = 0, 2$.

$$
\begin{aligned}
&\partial_\mu C^{(2),0}_{mat} - 4\psi_1 (e_{0\mu})^{\alpha\beta} C^{(2),2}_{mat\ \alpha\beta} = U^0_\mu, \\
&\nabla_\mu C^{(2),2}_{mat\ \alpha\beta} - 2\psi_1 (e_{0\mu})_{\alpha\beta} C^{(2),0}_{mat} - 24\psi_1 (e_{0\mu})^{\gamma\delta} C^{(2),4}_{mat\ \gamma\delta\alpha\beta} = U^2_{\mu|\alpha\beta},
\end{aligned}
\tag{2.69}
$$

where U^0_μ and $U^2_{\mu|\alpha_1\alpha_2}$ are the first two coefficient of the y-expansion of the RHS of (2.68). After some simple manipulations, it follows that

$$
(\Box - m^2) C^{(2),0}_{mat} = \nabla_\mu U^{0,\mu} + 4\psi_1 (e^\mu_0)^{\alpha\beta} U^2_{\mu|\alpha\beta}.
\tag{2.70}
$$

The RHS is calculated in terms of the first order fields in Appendix 2.B.2. The resulting the second order equation for the scalars can be written in the form

$$(\Box - m^2)C_{mat}^{(2),0} = \sum_{s=2}^{\infty} C_{mat}^{(1),2s-2}(\partial_y)\Xi_s(y), \tag{2.71}$$

where $\Xi_s(y)$ is expressed in terms of the higher spin fields as

$$\begin{aligned}\Xi_s(y) = 8&\left[\chi_{odd}^{(s),+}(y) + (2s-2)(2s-1)\chi_{odd}^{(s),-}(y)\right]\\ &+ 32\psi_1\left[\frac{1}{(2s-1)}\nabla^-\chi_{odd}^{(s),+}(y) - (2s-2)\nabla^+\chi_{odd}^{(s),-}(y)\right].\end{aligned} \tag{2.72}$$

2.4.2 The three point function

The boundary-to-bulk propagator for the higher spin gauge field satisfying the generalized Brown-Henneaux boundary condition (2.51) is determined by the boundary behavior of the gauge transformation (2.54). The latter is enough for us to compute the three point function of one higher spin gauge field and two scalars. Suppose the cubic action of a higher spin gauge field and two scalars is of the form as the higher spin gauge field couples to the higher spin current, i.e.

$$\int d^2x \left(\frac{dz}{z^3}\right) \Phi^s_{\mu_1\cdots\mu_s} T_s^{\mu_1\cdots\mu_s} \tag{2.73}$$

where the higher spin current $T_s^{\mu_1\cdots\mu_s}$ is a quadratic function of the scalar and its derivatives. Since the boundary to bulk propagator for high spin gauge field can be written in a "pure gauge" form: $\Phi^s_{\mu_1\cdots\mu_s} = \nabla_{(\underline{\mu_1}}\eta^s_{\underline{\mu_2\cdots\mu_s})}$, and the higher spin current is conserved: $\nabla_\mu T_s^{\mu\mu_1\cdots\mu_{s-1}} = 0$, we have

$$\begin{aligned}&\int d^2x \left(\frac{dz}{z^3}\right) \nabla_{\mu_1}\eta^s_{\mu_2\cdots\mu_s} T_s^{\mu_1\cdots\mu_s}\\ &= \int d^2x dz \partial_{\mu_1}\left(\frac{1}{z^3}\eta^s_{\mu_2\cdots\mu_s} T_s^{\mu_1\cdots\mu_s}\right)\\ &= -\lim_{z\to 0}\frac{1}{z^3}\int d^2x\, \eta^s_{\mu_2\cdots\mu_s} T_s^{z\mu_2\cdots\mu_s},\end{aligned} \tag{2.74}$$

which only depends on the boundary behavior of the gauge parameter at $z \to 0$.

The RHS of the second order equation (2.71) gives the variation of the cubic action with respect to the scalar up to some possible boundary terms.

$$\delta S = \int d\psi_1 \int \frac{d^2x dz}{z^3} \psi_1 \delta C^{(1),0}_{mat} \sum_{s=2}^{\infty} C^{(1),2s-2}_{mat}(\partial_y) \Xi_s(y). \tag{2.75}$$

While it is possible to recover the cubic part of the action from (2.75), in the form (2.73), we will not need it for the computation of the three point function. The tree level three point function is computed by varying the bulk action with respect to three sources inserted on the boundary, and so it suffices to work with (2.75) directly, by evaluating it on the boundary-to-bulk propagators for the higher spin gauge field and scalars. This computation is performed explicitly in Appendix 2.B.3. The resulting three point function of one higher spin current and two scalars is:

$$\left\langle \overline{\mathcal{O}}(x_1) \mathcal{O}(x_2) J_s(x_3) \right\rangle = -4\pi(s + \widetilde{\psi}_1(s-1))\Gamma(s) \frac{1}{|x_{12}|^{2+\widetilde{\psi}_1}} \left(\frac{x_{12}^-}{x_{13}^- x_{23}^-} \right)^s. \tag{2.76}$$

Here $\mathcal{O}$ and $\overline{\mathcal{O}}$ are dual to $C_{even} + C_{odd}$ and $C_{even} - C_{odd}$ respectively. They have scaling dimension $\Delta_+ = \frac{3}{2}$ or $\Delta_- = \frac{1}{2}$ depending on the choice of boundary condition, corresponding to $\widetilde{\psi}_1 = 1$ or $\widetilde{\psi}_1 = -1$. The position dependent factor on the RHS of (2.76) is fixed by conformal symmetry. The only nontrivial data here are contained in the overall coefficient, which is unambiguous given the normalization of the currents. These will be compared to representations of the W_N algebra in the 't Hooft limit in the next section.

2.5 The dual CFT

2.5.1 The proposal

It has been proposed in [4] that Vasiliev's higher spin-matter system (more precisely, a version of this theory with four real massive scalars) is dual to the W_N minimal model, which can be realized by the coset model

$$\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}}. \tag{2.77}$$

This CFT has a 't Hooft-like scaling limit, in which N is taken to be large while keeping the 't Hooft coupling

$$\lambda = \frac{N}{N+k} \tag{2.78}$$

to be fixed. In the infinite N limit, λ becomes a continuous parameter, in the range $0 < \lambda < 1$. It is proposed that λ is mapped to the parameter ν that label AdS_3 vacua, with the identification $\lambda = \frac{1}{2}(1 \pm \nu)$. The undeformed, $\nu = 0$ vacuum we have been considering so far would be mapped to the $\lambda = 1/2$ case. In the 't Hooft limit, "basic primaries" of (left plus right) scaling dimension $\Delta_\pm = 1 \pm \lambda$ are mapped to the massive scalars in the bulk, whereas all other primaries are found in the OPEs of the basic primaries, their duals interpreted as bound states in the bulk.

A puzzle with this proposal is the existence of low lying primary operators in the coset CFT, whose dimension scale like $1/N$ and form a discretuum in the 't Hooft limit. This has been further addressed in [30]. It is unclear how to interpret the dual of such states in the bulk.

Here we put forward a different proposal, namely that the Vasiliev higher spin-matter system, involving only two real massive scalars in the bulk, is dual to a subsector of the W_N

minimal model, generated by the two basic primaries of either dimension Δ_+ or dimension Δ_-, depending on the boundary condition for the bulk scalar field. This subsector has closed OPE and is consistent as a CFT on the sphere, though not on Riemann surfaces of nonzero genus, as it is not modular invariant. Hence, we believe that the bulk Vasiliev's system is nonperturbatively incomplete, though makes sense perturbatively to all orders in its coupling constant (i.e. $1/N$).

In a similar manner, we further propose that the "minimal" Valisiev's system, obtained via the truncation to fields invariant under the ι-involution (2.8), is dual to a subsector of the orthogonal group version of the coset model,[9]

$$\frac{SO(N)_k \oplus SO(N)_1}{SO(N)_{k+1}}. \tag{2.79}$$

Because $SO(N)$ has only even degree Casimir invariants, the coset model contains only the even spin currents. The real scalar in the "minimal" Valisiev's system is dual to one of the real basic primary operators, either $(\square; 0)$ or $(0; \square)$, depending on the choice of boundary condition for the bulk scalar.

2.5.2 W_N currents and primaries

Let $K^a(z)$ be the currents of the $SU(N)_k$ current algebra, and $J^a(z)$ the currents of $SU(N)_1$. Our convention for the group generators of $SU(N)$ is such that

$$\mathrm{Tr}(T^a T^b) = -\delta^{ab} \tag{2.80}$$

[9]The bulk gauge group of the minimal Vasiliev theory, in the Chern-Simons language, when truncated to a finite (even) spin N, is $Sp(N, \mathbb{R}) \times Sp(N, \mathbb{R})$. In mapping representations of the higher spin algebra in the bulk to primaries labeled by representations of the affine Lie algebra of the minimal model, a transpose on the Young tableaux is involved [30]. This suggests that the dual minimal model is based on SO rather than Sp coset. We thank T. Hartman for pointing this out. Note also that the analogous Sp coset construction would not give a W_N minimal model; its primaries are generally not labelled simply by a pair of representations, but a triple of representations [31].

where Tr is taken in the fundamental representation. The cubic symmetric tensor is defined to be

$$d^{abc} = -i\mathrm{Tr}(\{T^a, T^b\}T^c). \tag{2.81}$$

The $SU(N)_k$ currents, for instance, are normalized with the OPE

$$K^a(z)K^b(0) \sim -\frac{k}{z^2}\delta^{ab} + f^{abc}\frac{K^c(0)}{z}, \tag{2.82}$$

where $f^{abc} = -\mathrm{Tr}([T^a, T^b]T^c)$. The spin-2 current, i.e. the stress-energy tensor of the coset model constructed out of the Sugawara tensors, is given by

$$\begin{aligned} T(z) &= W^2(z) \\ &= -\frac{1}{2(N+k)} : K^a K^a : -\frac{1}{2(N+1)} : J^a J^a : +\frac{1}{2(N+k+1)} : (K^a + J^a)(K^a + J^a) : \end{aligned} \tag{2.83}$$

The spin-3 current W^3, in the 't Hooft limit, is written as

$$W^3(z) = d_{abc}\left[\frac{3\lambda^2}{(1-\lambda)(2-\lambda)} : K^a K^b J^c : -\frac{3\lambda}{1-\lambda} : K^a J^b J^c : + : J^a J^b J^c :\right]. \tag{2.84}$$

The normalization is such that the two point function of W^3 is given by

$$\langle W^3(z)W^3(0)\rangle = -6\frac{(1+\lambda)(2+\lambda)}{(1-\lambda)(2-\lambda)}N^5 + (1/N \text{ corrections}). \tag{2.85}$$

One may also construct higher spin-s currents out of the product of s K^a and J^a's, subject to the constraint that W^s is primary with respect to the diagonal $SU(N)_{k+1}$. This is rather cumbersome, which we shall not attempt here. Nonetheless, we will perform one unambiguous check with the spin-3 current.

Let us now turn to the primary operators with respect to the W_N algebra. These are labelled by three representations of $SU(N)$, $(\rho, \mu; \nu)$; here ρ, μ, ν are the height weight vectors of the respective representations, subject to the condition that the sum of the Dynkin labels is less than or equal to the level, and the constraint that $\rho + \mu - \nu$ lies in the root lattice of $SU(N)$. Further, it follows from the second $SU(N)$ being at level 1 that μ is uniquely determined given ρ and ν. Following the notation of [4], the primaries are labeled by $(\rho;\nu)$. We consider the diagonal modular invariant, by pairing up identical representations on the left and right moving sectors. The basic primaries are:

$$\begin{aligned} \mathcal{O}_+ &= (\Box;0)\otimes(\Box;0), \qquad \overline{\mathcal{O}}_+ = (\overline{\Box};0)\otimes(\overline{\Box};0), \\ \mathcal{O}_- &= (0;\Box)\otimes(0;\Box), \qquad \overline{\mathcal{O}}_- = (0;\overline{\Box})\otimes(0;\overline{\Box}). \end{aligned} \tag{2.86}$$

In the 't Hooft limit, $\mathcal{O}_\pm$ (and $\overline{\mathcal{O}}_\pm$) have conformal weight $h_\pm = \bar{h}_\pm = \frac{1\pm\lambda}{2}$.

Our proposal is that with the Δ_+ boundary condition, the two real massive scalars in the bulk, combined into a complex scalar $C_{even} + C_{odd}$, is dual to $\mathcal{O}_+$, while its complex conjugate $C_{even} - C_{odd}$ is dual to $\overline{\mathcal{O}}_+$. According to the fusion rule, the OPEs of $\mathcal{O}_+$ and $\overline{\mathcal{O}}_+$ involve only primaries labeled by the representations of the form $(R;0)$. In particular, the operators $\mathcal{O}_-, \overline{\mathcal{O}}_-$ and the low lying primaries of the form $(R;R)$ do not appear in the OPEs of $\mathcal{O}_+$ and $\overline{\mathcal{O}}_+$. Thus, this subsector of the CFT closes on the sphere.

Alternatively, with Δ_- boundary condition imposed on the bulk scalar, we propose the dual to the be subsector generated by $\mathcal{O}_-$ and $\overline{\mathcal{O}}_-$.

2.5.3 A test on the three point function

The spin-3 current acts on the basic primaries $\mathcal{O}_\pm$ as

$$\begin{aligned} W_0^3|\mathcal{O}_-\rangle &= C_\Box|\mathcal{O}_-\rangle, \\ W_0^3|\mathcal{O}_+\rangle &= -C_\Box\frac{(1+\lambda)(2+\lambda)}{(1-\lambda)(2-\lambda)}|\mathcal{O}_+\rangle, \end{aligned} \tag{2.87}$$

where $C_\Box$ is the cubic Casimir for the fundamental representation, given by

$$C_\Box|\Box\rangle = d_{abc}J_0^a J_0^b J_0^c|\Box\rangle, \qquad C_\Box = iN^2 \tag{2.88}$$

in our convention. The three point function $\langle \mathcal{O}_\Delta(z_1)\mathcal{O}_\Delta(z_2)W^s(z_3)\rangle$ is determined by conformal symmetry to be of the form

$$\frac{A(s)}{|z_{12}|^{2\Delta}}\left(\frac{z_{12}}{z_{13}z_{23}}\right)^s. \tag{2.89}$$

We will write $\langle \overline{\mathcal{O}}_\Delta \mathcal{O}_\Delta W^s\rangle \equiv A(s)$ for the coefficient. It follows from the action of W_0^3 on the primary states that

$$\langle \overline{\mathcal{O}}_+\mathcal{O}_+W^3\rangle = -iN^2\frac{(1+\lambda)(2+\lambda)}{(1-\lambda)(2-\lambda)}, \qquad \langle \overline{\mathcal{O}}_-\mathcal{O}_-W^3\rangle = iN^2. \tag{2.90}$$

If we define $J^{(s)}$ to be the spin-s current with normalized two-point function, namely $\langle J^{(s)}(z)J^{(s)}(0)\rangle = z^{-2s}$ (this fixes $J^{(s)}$ up to a sign), then we have

$$\begin{aligned}
&\langle \overline{\mathcal{O}}_+\mathcal{O}_+J^{(2)}\rangle = N^{-\frac{1}{2}}\sqrt{\frac{1+\lambda}{2(1-\lambda)}}, \qquad \langle \overline{\mathcal{O}}_-\mathcal{O}_-J^{(2)}\rangle = N^{-\frac{1}{2}}\sqrt{\frac{1-\lambda}{2(1+\lambda)}},\\
&\langle \overline{\mathcal{O}}_+\mathcal{O}_+J^{(3)}\rangle = N^{-\frac{1}{2}}\sqrt{\frac{(1+\lambda)(2+\lambda)}{6(1-\lambda)(2-\lambda)}}, \qquad \langle \overline{\mathcal{O}}_-\mathcal{O}_-J^{(3)}\rangle = -N^{-\frac{1}{2}}\sqrt{\frac{(1-\lambda)(2-\lambda)}{6(1+\lambda)(2+\lambda)}}.
\end{aligned} \tag{2.91}$$

From the bulk, we have computed the three point function $\langle \overline{\mathcal{O}}\mathcal{O}J^{(s)}\rangle$ in the undeformed theory, with the result (after normalizing the spin-s current)

$$\langle \overline{\mathcal{O}}_+\mathcal{O}_+J^{(s)}\rangle = g\Gamma(s)\sqrt{\frac{2s-1}{\Gamma(2s-1)}}, \qquad \langle \overline{\mathcal{O}}_-\mathcal{O}_-J^{(s)}\rangle = (-)^s g\frac{\Gamma(s)}{\sqrt{\Gamma(2s)}}. \tag{2.92}$$

Here g is the overall coupling constant of the bulk theory. This should be compared with the CFT at $\lambda = 1/2$. With the identification

$$g = \frac{1}{\sqrt{N}}, \tag{2.93}$$

we see that (2.92) precisely agrees with (2.91) at $\lambda = 1/2$. (2.92) then further makes predictions for the three point functions $\langle \overline{\mathcal{O}}\mathcal{O}J^{(s)}\rangle$ of spin $s \geq 4$ in the W_N coset CFT, in the 't Hooft limit at $\lambda = 1/2$, which remains to be computed directly on the CFT side.

Further, it would be very interesting to compute these three point functions in the *deformed* bulk theory, i.e. the AdS$_3$ vacua with nonzero ν, which should be mapped to the CFT with 't Hooft parameter away from $\lambda = 1/2$. We hope to report on this in future works.

2.6 Concluding remarks

In this paper, we have developed the perturbation theory of Vasiliev's higher spin-matter system in AdS$_3$, to the second order. This allowed us to compute the bulk tree level three point functions, in the undeformed $\nu = 0$ vacuum. The result passed a nontrivial test that involves the explicit expression for the spin-3 current in the W_N minimal model (at the special value of 't Hooft coupling $\lambda = 1/2$). Our result from the bulk also makes predictions on three point functions involving currents of spin $s \geq 4$ which in principle can be straightforwardly computed (though tedious) in the coset CFT, by constructing the W_N currents out of the spin 1 affine currents, and then taking the 't Hooft limit.

A natural next step is to move away from the undeformed, $\nu = 0$ vacuum, and consider the deformed bulk theory, which should be dual to the CFT away from $\lambda = 1/2$. In Appendix 2.C, we have derived the boundary to bulk propagator for the scalar master field in the deformed theory. The computation of correlators using these expressions could be complicated, though at least one can work order by order expanding in ν, which amounts to expanding in $\lambda - \frac{1}{2}$ in the dual CFT.

Next, one would like to go beyond leading order in $1/N$. The basic primaries in the W_N

minimal model have exact scaling dimensions

$$\begin{aligned} \Delta_+ &= 2h(\Box;0) = \frac{N-1}{N}(1+\frac{N+1}{N}\lambda), \\ \Delta_- &= 2h(0;\Box) = \frac{N-1}{N}(1-\frac{N+1}{N+\lambda}\lambda). \end{aligned} \tag{2.94}$$

Identifying $\Delta_\pm = 1 \pm \sqrt{1+m_\pm^2}$, we see that the renormalized mass of the bulk scalar with the two different boundary conditions are

$$\begin{aligned} m_+^2 &= -\left[(1+\frac{\lambda}{N})^2 - \lambda^2\right]\left(1-\frac{1}{N^2}\right), \\ m_-^2 &= -(1-\lambda^2)\left(1+\frac{\lambda}{N}\right)^{-2}\left(1-\frac{1}{N^2}\right). \end{aligned} \tag{2.95}$$

The bulk scalar propagator depend on the boundary condition (Δ_+ or Δ_-), which presumably leads to the different renormalized masses m_+ and m_- through loop corrections. The difference between m_+ and m_-, say at order $1/N$, or one-loop in the bulk, can in principle be understood [32, 23] in terms of the tree level four-point functions, by factorizing the difference in the bulk propagators for the two boundary conditions into the product of boundary-to-bulk propagators. To compute either m_-^2 or m_+^2 form the bulk, however, requires performing a genuine one-loop computation in Vasiliev's theory. The precise relation between the bulk deformation parameter ν and the 't Hooft coupling λ of the boundary CFT, beyond the leading order in $1/N$, is presumably also regularization dependent.

We proposed that Vasiliev's system is dual to not the entire W_N minimal model CFT, but only a subsector of it, generated by the basic primaries $\mathcal{O}_+, \overline{\mathcal{O}}_+$ and the W_N currents, or the subsector generated by $\mathcal{O}_-, \overline{\mathcal{O}}_-$ and the W_N currents, depending on whether Δ_+ or Δ_- boundary condition is imposed on the two bulk scalars. These two subsectors close on their OPEs, and lead to consistent n-point functions on the sphere. However, they are not modular invariant. From the perspective of the bulk higher spin gravity theory, modular invariance is expected to be restored by gravitational instantons (analytic continuation of BTZ black holes), which are non-perturbative. At the level of perturbation theory, it is consistent that the bulk theory is dual to a subsector of a modular invariant CFT. The duality we are proposing is analogous to the statement that pure gravity in AdS$_3$, at the level of perturbation theory, is dual to the subsector of a CFT involving only Virasoro descendants of the vacuum, i.e. operators made out of products of stress-energy tensors. The latter lead to a consistent set of n-point functions on the sphere, though do not give modular invariant genus one partition functions by themselves.

If our proposal is correct, then it suggests that Vasiliev's system is non-perturbatively incomplete, though makes sense to all orders in perturbation theory. One may suspect that solitons, in particular black hole solutions, should be included and could make the theory modular invariant. However, we are not aware of a modular invariant completion of the Δ_+ or Δ_- subsector of W_N minimal model that requires adding only states/operators whose dimensions scale with N (and are large in the large N limit). The W_N minimal model itself would amount to adding not only states of dimension of order 1, but also a large number of light states whose dimensions go like $1/N$, which seems pathological from the perspective of the bulk theory.

It is clearly of great interest, still, to understand the bulk theory dual to the full W_N minimal model, since the latter is non-perturbative defined and exactly solvable. It is shown in [30] that the descendants of the light states give rise to bound states of the basic primaries, while the light states themselves become null in the infinite N limit. It is unclear how to understand this from the bulk. A possibility is that additional *massless* scalars should be added in the bulk theory, with the non-standard boundary condition (so that they are dual to operators of dimension 0 rather than 2, classically). It would be an interesting challenge to construct such a theory in AdS_3.

2.A Linearizing Vasiliev's equations

2.A.1 Derivation of the scalar boundary to bulk propagator

In this subsection, we study the linearized equations (2.18), and solve for the boundary-to-bulk propagator for the master field $C^{(1)}$.

Decomposing the $C^{(1)}$ as in (2.20) the equation $D_0 C^{(1)} = 0$ is written as

$$\begin{aligned} &d_x C^{(1)}_{aux} + 4(w_0^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \psi_1 e_0^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta}) C^{(1)}_{aux} = 0 \\ &d_x C^{(1)}_{mat} + 4 w_0^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} C^{(1)}_{mat} - 2\psi_1 e_0^{\alpha\beta} (y_\alpha y_\beta + \frac{\partial^2}{\partial y^\alpha \partial y^\beta}) C^{(1)}_{mat} = 0 \end{aligned} \tag{2.96}$$

Expand $C^{(1)}_{mat/aux}(x|y,\psi_i)$ as in (2.21), we write the first equation of (2.96) as

$$\partial_\mu C^{(1),n}_{aux\ \alpha_1\cdots\alpha_n} - 4n(w_{0\mu})_{(\underline{\alpha_1}}{}^{\beta} C^{(1),n}_{aux\ \beta\underline{\alpha_2\cdots\alpha_n})} - 4n\psi_1 (e_{0\mu})_{(\underline{\alpha_1}}{}^{\beta} C^{(1),n}_{aux\ \beta\underline{\alpha_2\cdots\alpha_n})} = 0. \tag{2.97}$$

Contracting this equation with $(e_0^\mu)_{\gamma\delta}$, and symmetrizing the indices $(\gamma\delta\alpha_1\cdots\alpha_n)$, we get

$$\nabla_{(\underline{\gamma\delta}} C^{(1),n}_{aux\ \underline{\alpha_1\cdots\alpha_n})} = 0 \quad \text{with} \quad \nabla_{\alpha\beta} = e^\mu_{\alpha\beta} \nabla_\mu, \tag{2.98}$$

which means that $C^{(1)}_{aux}$ carries no propagating degree of freedom. We can simply set $C^{(1)}_{aux} = 0$.

The second equation of (2.96) can be written as

$$\begin{aligned} &\partial_\mu C^{(1),n}_{mat\ \alpha_1\cdots\alpha_n} - 4n(w_{0\mu})_{(\underline{\alpha_1}}{}^{\beta} C^{(1),n}_{mat\ \beta\underline{\alpha_2\cdots\alpha_n})} \\ &\quad - 2\psi_1 (e_{0\mu})_{(\underline{\alpha_1\alpha_2}} C^{(1),n-2}_{mat\ \underline{\alpha_3\cdots\alpha_n})} - 2(n+2)(n+1)\psi_1 (e_{0\mu})^{\alpha\beta} C^{(1),n+2}_{mat\ \alpha\beta\alpha_1\cdots\alpha_n} = 0. \end{aligned} \tag{2.99}$$

Or contracting this equation with $(e_0^\mu)_{\alpha\beta}$ gives

$$\begin{aligned} &\nabla_{\alpha\beta} C^{(1),n}_{mat\ \alpha_1\cdots\alpha_n} + \frac{1}{16}\psi_1 \epsilon_{(\alpha(\underline{\alpha_1}}\epsilon_{\beta)\underline{\alpha_2}} C^{(1),n-2}_{mat\ \underline{\alpha_3\cdots\alpha_n})} \\ &\quad + \frac{1}{16}(n+2)(n+1)\psi_1 C^{(1),n+2}_{mat\ \alpha\beta\alpha_1\cdots\alpha_n} = 0. \end{aligned} \tag{2.100}$$

This equation is in a reducible representation of the permutation group of permuting the indices. To simplify the equation, we decompose it into irreducible representations by contracting with the tensor $\epsilon^{\alpha\beta}$ or symmetrizing all the indices. First, contracting (2.100) with $\epsilon^{\alpha\alpha_1}$ gives

$$\nabla^{\alpha}{}_{\beta} C^{(1),n}_{mat\ \alpha\alpha_2\cdots\alpha_n} - \frac{n+1}{16n}\psi_1 \epsilon_{\beta(\underline{\alpha_2}} C^{(1),n-2}_{mat\ \ \ \ \underline{\alpha_3\cdots\alpha_n})} = 0. \tag{2.101}$$

Contracting (2.101) with $\epsilon^{\beta\alpha_2}$ gives

$$\nabla^{\alpha\beta} C^{(1),n}_{mat\ \alpha\beta\alpha_3\cdots\alpha_n} + \frac{n+1}{16(n-1)}\psi_1 C^{(1),n-2}_{mat\ \ \ \ \alpha_3\cdots\alpha_n} = 0. \tag{2.102}$$

Next, we want to symmetrize the indices of equations (2.100), (2.101), and (2.102). It is convenient to reintroduce the auxiliary y^α-variable. By contracting the indices of the equations (2.100), (2.101), and (2.102) with the y^α's which automatically symmetrizes all the indices, we obtain

$$\begin{aligned}
&\nabla^+ C^{(1),n}_{mat}(y) - \frac{1}{16}(n+2)(n+1)\psi_1 C^{(1),n+2}_{mat}(y) = 0,\\
&\nabla^0 C^{(1),n}_{mat}(y) = 0,\\
&\nabla^- C^{(1),n}_{mat}(y) - \frac{1}{16}(n+1)n\psi_1 C^{(1),n-2}_{mat}(y) = 0,
\end{aligned} \tag{2.103}$$

where

$$C^{(1),n}_{mat}(y) = C^{(1),n}_{mat\ \alpha_1\cdots\alpha_n} y^{\alpha_1}\cdots y^{\alpha_n} \tag{2.104}$$

which is the degree n homogeneous polynomial in the Taylar expansion of the matter field $C^{mat}(y)$, and we define the operators

$$\nabla^+ = (y \not\nabla y), \qquad \nabla^0 = (y \not\nabla \partial_y), \qquad \nabla^- = (\partial_y \not\nabla \partial_y). \tag{2.105}$$

They obey commutation relations

$$\begin{aligned}
&[\nabla^0, \nabla^\pm] = 0, \\
&[\nabla^+, \nabla^-] = \frac{\mathcal{N}+1}{16}\Box_{AdS} - \frac{\mathcal{N}(\mathcal{N}+2)(\mathcal{N}+1)}{64}, \\
&(\nabla^0)^2 = \nabla^+\nabla^- + \frac{\mathcal{N}^2}{64}\Box_{AdS} + \frac{\mathcal{N}^2(\mathcal{N}+2)}{128}.
\end{aligned} \tag{2.106}$$

with $\mathcal{N} = y\partial_y$ and $\Box_{AdS} \equiv -32\nabla_{\alpha\beta}\nabla^{\alpha\beta}$ where $\nabla_{\alpha\beta}$ is defined to act covariantly both on explicit spinor indices as well as on indices contracted with y^α. Iterating the first equation of (2.103), we get

$$C^{(1),2s}_{mat}(y) = \frac{1}{(2s)!}(16\psi_1\nabla^+)^s C^{(1),0}_{mat}. \tag{2.107}$$

Since $C^{(1)}_{mat}(y)$ is an even function in y^α, it is totally determined by its lowest component $C^{(1),0}_{mat}$ via the above relation. After some simple manipulations of (2.103) using (2.106), we derive

$$\Box_{AdS} C^{(1),n}_{mat} = -\frac{1}{4}(3 + n(n+2))C^{(1),n}_{mat}. \tag{2.108}$$

For $n = 0$, the equation gives the usual Klein-Gordon equation on AdS$_3$, (2.22). The higher components $C^{(1),n}_{mat}$ are determined by $C^{(1),0}_{mat}$ through the linearized equations of motion.

The equation (2.22) is solved by scalar boundary to bulk propagator $C^{mat,0} = K(x,z)^\Delta$ for $\Delta = 3/2$ or $\Delta = 1/2$, where $K(x,z) \equiv \frac{z}{x^2+z^2}$. It is convenient to introduce another auxiliary variable $\tilde{\psi}_1$, satisfying $\tilde{\psi}_1^2 = 1$, to label the different boundary conditions, so that $\Delta = 1 + \tilde{\psi}_1/2$. The $(\nabla^+)^s$ acting on K^Δ is

$$(\nabla^+)^s K^\Delta = \frac{1}{8^s}\left(\prod_{j=1}^{s}(\Delta + j - 1)\right)(y\Sigma y)^s K^\Delta, \tag{2.109}$$

and using (2.107), we obtain

$$C^{(1)}_{mat}(y) = \left(1 + \psi_1 \frac{1+\tilde{\psi}_1}{2} y\Sigma y\right) e^{\frac{\psi_1}{2} y\Sigma y} K^{1+\frac{\tilde{\psi}_1}{2}}, \tag{2.110}$$

where $\Sigma = \sigma^z - \frac{2z}{x^2}\sigma^\mu x^\mu$.

2.A.2 The linearized higher spin equations

In this subsection, we study the linearized equations (2.14),(2.15),(2.16), and rewrite them as the (linearized) Chern-Simons equation and Fronsdal equation by eliminating all the auxiliary degrees of freedom.

The (2.15) and (2.16) imply that W' is solved in terms of S and further in terms of $C^{(1)}_{mat}$; hence, in particular, it is linear in ψ_2. Decomposing $\Omega^{(1)}$ as in (2.28), the linearized equations are written in (2.29).

The linearized gauge transformations act by

$$\begin{aligned}\delta W^{(1)} &= d_x\epsilon + [W_0, \epsilon]_*,\\ \delta S^{(1)} &= d_z\epsilon.\end{aligned} \tag{2.111}$$

Let us restrict to gauge transformations that leave $S^{(1)}$ invariant, namely $\epsilon = \lambda(x|y,\psi_1) + \psi_2\rho(x|y,\psi_1)$, where $\lambda(x|y,\psi_1)$ and $\rho(x|y,\psi_1)$ transform Ω^{hs} and Ω^{sc} independently at the linearized level. Their actions are

$$\begin{aligned}\delta\Omega^{sc} &= d_x\rho + \psi_2[W_0, \psi_2\rho]_* = \nabla_x\rho - \psi_1\{e_0, \rho\}_*,\\ \delta\Omega^{hs} &= d_x\lambda + [W_0, \lambda]_* = \nabla_x\lambda + \psi_1[e_0, \lambda]_*.\end{aligned} \tag{2.112}$$

We show that Ω^{sc} contains no dynamical degrees of freedom. First consider the homogeneous part of the equation,

$$\tilde{D}_0\Omega^{sc} = 0, \tag{2.113}$$

or more explicitly,

$$\nabla_x\Omega^{sc}(x|y,\psi_1) - \psi_1 e_0(x|y) \wedge_* \Omega^{sc}(x|y,\psi_1) + \psi_1\Omega^{sc}(x|y,\psi_1) \wedge_* e_0(x|y) = 0. \tag{2.114}$$

We have emphasized the wedge product between 1-forms, so the last terms involve the

$*$-anti-commutator of the components of e_0 and Ω^{sc}. Expand Ω^{sc} as

$$\Omega^{sc}(x|y,\psi_1) = dx^\mu \sum_{n=0}^{\infty} \Omega^{sc,n}_{\mu|\alpha_1\cdots\alpha_n}(x|\psi_1) y^{\alpha_1}\cdots y^{\alpha_n}. \tag{2.115}$$

In components, the homogeneous equation for Ω^{sc} is written as

$$\nabla_{[\mu}\Omega^{sc,n}_{\nu]|\alpha_1\cdots\alpha_n} - 2\psi_1(e_{0[\mu})_{(\underline{\alpha_1\alpha_2}}\Omega^{sc,n-2}_{\nu]|\underline{\alpha_3\cdots\alpha_n})} - 2(n+2)(n+1)\psi_1(e_{0[\mu})^{\alpha\beta}\Omega^{sc,n+2}_{\nu]|\alpha\beta\alpha_1\cdots\alpha_n} = 0. \tag{2.116}$$

Converting μ,ν into spinor indices, we obtain

$$\nabla_{(\alpha}{}^{\gamma}\Omega^{sc,n}_{\beta)\gamma|\alpha_1\cdots\alpha_n} - 2\psi_1 e_{\alpha}{}^{\gamma}{}_{|(\underline{\alpha_1\alpha_2}}\Omega^{sc,n-2}_{\beta)\gamma|\underline{\alpha_3\cdots\alpha_n})} - 2(n+2)(n+1)\psi_1 e_{(\alpha}{}^{\gamma|\delta\tau}\Omega^{sc,n+2}_{\beta)\gamma|\delta\tau\alpha_1\cdots\alpha_n} = 0. \tag{2.117}$$

where

$$e_{\alpha\beta|\gamma\delta} \equiv (e_0^\mu)_{\alpha\beta}(e_{0\mu})_{\gamma\delta} = -\frac{1}{64}(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}). \tag{2.118}$$

We can write (2.117) as

$$\nabla_{(\alpha}{}^{\gamma}\Omega^{sc,n}_{\beta)\gamma|\alpha_1\cdots\alpha_n} - \frac{1}{16}\psi_1\epsilon_{(\alpha(\underline{\alpha_1}}\Omega^{sc,n-2}_{\beta)\underline{\alpha_2}|\underline{\alpha_3\cdots\alpha_n})} + \frac{1}{16}(n+2)(n+1)\psi_1\epsilon^{\gamma\delta}\Omega^{sc,n+2}_{\gamma(\alpha|\beta)\delta\alpha_1\cdots\alpha_n} = 0. \tag{2.119}$$

In components, the gauge transformation (2.112) for Ω^{sc} can be written as

$$\delta\Omega^{sc,n}_{\mu|\alpha_1\cdots\alpha_n} = \nabla_\mu\rho^n_{\alpha_1\cdots\alpha_n} - 2\psi_1(e_\mu)_{(\alpha_1\alpha_2}\rho^{n-2}_{\alpha_3\cdots\alpha_n)} - 2(n+2)(n+1)\psi_1(e_\mu)^{\alpha\beta}\rho^{n+2}_{\alpha\beta\alpha_1\cdots\alpha_n}, \tag{2.120}$$

or

$$\delta\Omega^{sc,n}_{\alpha\beta|\alpha_1\cdots\alpha_n} = \nabla_{\alpha\beta}\rho^n_{\alpha_1\cdots\alpha_n} + \frac{1}{16}\psi_1\epsilon_{(\alpha(\underline{\alpha_1}}\epsilon_{\beta)\underline{\alpha_2}}\rho^{n-2}_{\underline{\alpha_3\cdots\alpha_n})} + \frac{1}{16}(n+2)(n+1)\psi_1\rho^{n+2}_{\alpha\beta\alpha_1\cdots\alpha_n}. \tag{2.121}$$

Decomposing $\Omega^{sc,(n)}_{\alpha\beta|\alpha_1\cdots\alpha_n}$ as

$$\Omega^{sc,(n)}_{\alpha\beta|\alpha_1\cdots\alpha_n} = \zeta^{n,+}_{\alpha\beta\alpha_1\cdots\alpha_n} + \epsilon_{(\alpha_1(\underline{\alpha}}\zeta^{n,0}_{\underline{\beta})\alpha_2\cdots\alpha_n)} + \epsilon_{(\underline{\alpha}(\alpha_1}\epsilon_{\underline{\beta})\alpha_2}\zeta^{n,-}_{\alpha_3\cdots\alpha_n)}, \tag{2.122}$$

we find that $\zeta^{n,+}$ and $\zeta^{n,-}$ can be gauged away by ρ^{n+2} and ρ^{n-2}. Furthermore, by symmetrizing $(\alpha\beta\alpha_1\cdots\alpha_m)$ of (2.119), $\zeta^{n,0}$ can be fully determined by $\zeta^{n,+}$ and $\zeta^{n,-}$.

Now let us turn to the higher spin fields, Ω^{hs}. Their linearized equations are written more explicitly as

$$\nabla_x\Omega^{hs} + e_0 \wedge_* \Omega^{hs} + \Omega^{hs} \wedge_* e_0 = 0, \tag{2.123}$$

or in components,

$$\nabla_{[\mu}\Omega^{hs,n}_{\nu]|\alpha_1\cdots\alpha_n} - 4n\psi_1(e_{0[\mu})_{(\underline{\alpha_1}}{}^{\beta}\Omega^{hs,n}_{\nu]|\beta\underline{\alpha_2\cdots\alpha_n})} = 0. \tag{2.124}$$

Replacing $[\mu\nu]$ with spinor indices, we can write it as

$$\nabla_{(\alpha}{}^{\gamma}\Omega^{hs,n}_{\beta)\gamma|\alpha_1\cdots\alpha_n} - 4n\psi_1 e_{(\alpha}{}^{\gamma}{}_{|(\underline{\alpha_1}}{}^{\delta}\Omega^{hs,n}_{\beta)\gamma|\delta\underline{\alpha_2\cdots\alpha_n})} = 0, \tag{2.125}$$

or

$$\nabla_{(\alpha}{}^{\gamma}\Omega^{hs,n}_{\beta)\gamma|\alpha_1\cdots\alpha_n} + \frac{1}{16}n\psi_1\epsilon_{(\underline{\alpha_1}(\alpha}\Omega^{hs,n}{}_{\beta)}{}^{\gamma}{}_{|\gamma\underline{\alpha_2\cdots\alpha_n})} - \frac{1}{16}n\psi_1\Omega^{hs,n}_{(\alpha(\underline{\alpha_1}|\beta)\underline{\alpha_2\cdots\alpha_n})} = 0. \tag{2.126}$$

Let us decompose $\Omega^{hs,(n)}_{\alpha\beta|\alpha_1\cdots\alpha_n}$ into the irreducible representation of the permutation group of permuting the indices as

$$\Omega^{hs,(n)}_{\alpha\beta|\alpha_1\cdots\alpha_n} = \chi^{n,+}_{\alpha\beta\alpha_1\cdots\alpha_n} + \epsilon_{(\alpha_1(\underline{\alpha}}\chi^{n,0}_{\underline{\beta})\alpha_2\cdots\alpha_n)} + \epsilon_{(\underline{\alpha}(\alpha_1}\epsilon_{\underline{\beta})\alpha_2}\chi^{n,-}_{\alpha_3\cdots\alpha_n)}. \tag{2.127}$$

Conversely,

$$\begin{aligned}
&\Omega^{hs,n}_{(\alpha\beta|\alpha_1\cdots\alpha_n)} = \chi^{n,+}_{\alpha\beta\alpha_1\cdots\alpha_n},\\
&\Omega^{hs,n}{}_{(\alpha_1}{}^{\gamma}{}_{|\gamma\alpha_2\cdots\alpha_n)} = \frac{n+2}{2n}\chi^{n,0}_{\alpha_1\cdots\alpha_n},\\
&\Omega^{hs,n\gamma\delta}{}_{|\gamma\delta\alpha_1\cdots\alpha_{n-2}} = \frac{n+1}{n-1}\chi^{n,-}_{\alpha_1\cdots\alpha_{n-2}}.
\end{aligned} \tag{2.128}$$

Next, we want to also decompose the equation (2.126) into the irreducible representation of the permutation group. Symmetrizing all indices $(\alpha\beta\alpha_1\cdots\alpha_n)$ in (2.126) gives

$$\nabla_{(\alpha_1}{}^{\gamma}\chi^{n,+}_{\alpha_2\cdots\alpha_{n+2})\gamma} - \frac{1}{2}\nabla_{(\alpha_1\alpha_2}\chi^{n,0}_{\alpha_3\cdots\alpha_{n+2})} - \frac{1}{16}n\psi_1\chi^{n,+}_{\alpha_1\cdots\alpha_{n+2}} = 0. \tag{2.129}$$

On the other hand, contracting (2.126) with $\epsilon^{\alpha\alpha_1}$ gives

$$\begin{aligned}&\nabla_\alpha{}^\gamma\Omega_{\beta\gamma|}{}^\alpha{}_{\alpha_2\cdots\alpha_n}+\nabla_\beta{}^\gamma\Omega_{\alpha\gamma|}{}^\alpha{}_{\alpha_2\cdots\alpha_n}\\&-\frac{\psi_1}{16}\left[(n+3)\Omega_\beta{}^\gamma{}_{|\gamma\alpha_2\cdots\alpha_n}+(n-1)\epsilon_{(\underline{\alpha_2}\beta}\Omega^{\gamma\delta}{}_{|\gamma\delta\underline{\alpha_3\cdots\alpha_n})}+(n-1)\Omega_{\alpha(\underline{\alpha_2}|\beta}{}^\alpha{}_{\underline{\alpha_3\cdots\alpha_n})}\right]=0.\end{aligned}\tag{2.130}$$

Now symmetrizing $(\beta\alpha_2\cdots\alpha_n)$ gives

$$-\nabla^{\gamma\delta}\chi^{n,+}_{\gamma\delta\alpha_1\cdots\alpha_n}-\frac{2}{n}\nabla_{(\alpha_1}{}^\gamma\chi^{n,0}_{\alpha_2\cdots\alpha_n)\gamma}+\frac{n+2}{n}\nabla_{(\alpha_1\alpha_2}\chi^{n,-}_{\alpha_3\cdots\alpha_n)}-\frac{n+2}{8n}\psi_1\chi^{n,0}_{\alpha_1\cdots\alpha_n}=0.\tag{2.131}$$

Alternatively, contract (2.130) with $\epsilon^{\beta\alpha_2}$ gives

$$\frac{n+2}{n}\nabla^{\gamma\delta}\chi^{n,0}_{\gamma\delta\alpha_1\cdots\alpha_{n-2}}-\frac{2(n+1)(n-2)}{n(n-1)}\nabla^\gamma{}_{(\alpha_1}\chi^{n,-}_{\alpha_2\cdots\alpha_{n-2})\gamma}+\frac{(n+2)(n+1)}{8(n-1)}\psi_1\chi^{n,-}_{\alpha_1\cdots\alpha_{n-2}}=0.\tag{2.132}$$

As in the previous subsection, we reintroduce the auxiliary variable y^α, and define

$$\begin{aligned}\chi^+_n(y)&=\chi^{n,+}_{\alpha_1\cdots\alpha_{n+2}}y^{\alpha_1}\cdots y^{\alpha_{n+2}},\\\chi^0_n(y)&=\chi^{n,0}_{\alpha_1\cdots\alpha_n}y^{\alpha_1}\cdots y^{\alpha_n},\\\chi^-_n(y)&=\chi^{n,-}_{\alpha_1\cdots\alpha_{n-2}}y^{\alpha_1}\cdots y^{\alpha_{n-2}},\end{aligned}\tag{2.133}$$

and so

$$\Omega^{hs,(n)}_{\alpha\beta}(y)=\frac{1}{(n+2)(n+1)}\partial_\alpha\partial_\beta\chi^+_n(y)+\frac{1}{n}y_{(\alpha}\partial_{\beta)}\chi^0_n(y)+y_\alpha y_\beta\chi^-_n(y).\tag{2.134}$$

The three equations derived previously for χ, (2.129), (2.131), and (2.132), can now be written as

$$\begin{aligned}&\frac{1}{n+2}\nabla^0\chi^+_n(y)+\frac{1}{2}\nabla^+\chi^0_n(y)-\frac{n}{16}\psi_1\chi^+_n(y)=0,\\&\frac{1}{(n+2)(n+1)}\nabla^-\chi^+_n(y)-\frac{2}{n^2}\nabla^0\chi^0_n(y)-\frac{n+2}{n}\nabla^+\chi^-_n(y)-\frac{n+2}{8n}\psi_1\chi^0_n(y)=0,\\&-\frac{n+2}{n^2(n-1)}\nabla^-\chi^0_n(y)-\frac{2(n+1)}{n(n-1)}\nabla^0\chi^-_n(y)+\frac{(n+2)(n+1)}{8(n-1)}\psi_1\chi^-_n(y)=0.\end{aligned}\tag{2.135}$$

Now expand $\chi_n^{\pm/0}$ in ψ_1,

$$\chi_n^{\pm/0} = \chi_{even}^{n,\pm/0} + \psi_1 \chi_{odd}^{n,\pm/0}. \tag{2.136}$$

We can now solve χ_{even} in terms of χ_{odd}:

$$\begin{aligned}
\chi_{even}^{n,+}(y) &= \frac{16}{n}\left[\frac{1}{n+2}\nabla^0\chi_{odd}^{n,+}(y) + \frac{1}{2}\nabla^+\chi_{odd}^{n,0}(y)\right], \\
\chi_{even}^{n,0}(y) &= \frac{8}{n+2}\left[\frac{n}{(n+2)(n+1)}\nabla^-\chi_{odd}^{n,+}(y) - \frac{2}{n}\nabla^0\chi_{odd}^{n,0}(y) - (n+2)\nabla^+\chi_{odd}^{n,-}(y)\right], \\
\chi_{even}^{n,-}(y) &= \frac{8}{n}\left[\frac{1}{n(n+1)}\nabla^-\chi_{odd}^{n,0}(y) + \frac{2}{n+2}\nabla^0\chi_{odd}^{n,-}(y)\right].
\end{aligned} \tag{2.137}$$

At this point, it is convenient to use part of the gauge symmetry to gauge away χ_{odd}^0 completely (we will show this in the later part of this subsection), and then write

$$\begin{aligned}
\chi_{even}^{n,+}(y) &= \frac{16}{n(n+2)}\nabla^0\chi_{odd}^{n,+}(y), \\
\chi_{even}^{n,0}(y) &= \frac{8}{n+2}\left[\frac{n}{(n+2)(n+1)}\nabla^-\chi_{odd}^{n,+}(y) - (n+2)\nabla^+\chi_{odd}^{n,-}(y)\right], \\
\chi_{even}^{n,-}(y) &= \frac{16}{n(n+2)}\nabla^0\chi_{odd}^{n,-}(y).
\end{aligned} \tag{2.138}$$

Plugging back in (2.135) (with $\chi_{odd}^0 = 0$), we obtain (the second equation is automatically satisfied because of the second equation of (2.106))

$$\begin{aligned}
&\frac{16}{n(n+2)^2}(\nabla^0)^2\chi_{odd}^{n,+}(y) + \frac{4n}{(n+2)^2(n+1)}\nabla^+\nabla^-\chi_{odd}^{n,+}(y) - 4(\nabla^+)^2\chi_{odd}^{n,-}(y) - \frac{n}{16}\chi_{odd}^{n,+}(y) = 0, \\
&-\frac{8}{(n+2)(n+1)n}(\nabla^-)^2\chi_{odd}^{n,+}(y) + \frac{8(n+2)}{n^2}\nabla^-\nabla^+\chi_{odd}^{n,-}(y) - \frac{32(n+1)}{n^2(n+2)}(\nabla^0)^2\chi_{odd}^{n,-}(y) \\
&\quad + \frac{(n+2)(n+1)}{8}\chi_{odd}^{n,-}(y) = 0.
\end{aligned} \tag{2.139}$$

By using (2.106), we rewrite (2.139) as

$$\begin{aligned}
&\Box_{AdS}\chi_{odd}^{n,+}(y) + \frac{2n+8-n^2}{4}\chi_{odd}^{n,+}(y) + \frac{16}{(n+1)}\nabla^+\nabla^-\chi_{odd}^{n,+}(y) - 16n(\nabla^+)^2\chi_{odd}^{n,-}(y) = 0, \\
&\Box_{AdS}\chi_{odd}^{n,-}(y) - \frac{(n^2+2n+4)}{4}\chi_{odd}^{n,-}(y) - \frac{8}{n}\nabla^+\nabla^-\chi_{odd}^{n,-}(y) + \frac{8}{(n+1)n^2}(\nabla^-)^2\chi_{odd}^{n,+}(y) = 0.
\end{aligned} \tag{2.140}$$

Now let us examine the gauge transformations on $\chi^{\pm}$. The gauge transformation on the components of $\Omega^{hs,n}$ is

$$\delta\Omega^{hs,n}_{\alpha\beta|\alpha_1\cdots\alpha_n} = \nabla_{\alpha\beta}\lambda^n_{\alpha_1\cdots\alpha_n} - \frac{n}{16}\psi_1\epsilon_{(\alpha_1(\underline{\alpha}}\lambda^n_{\underline{\beta})\alpha_2\cdots\alpha_n)}. \tag{2.141}$$

In terms of $\chi^{\pm,0}$, we have

$$\begin{aligned}
\delta\chi^{n,+}_{\alpha_1\cdots\alpha_{n+2}} &= \nabla_{(\alpha_1\alpha_2}\lambda^n_{\alpha_3\cdots\alpha_{n+2})},\\
\delta\chi^{n,0}_{\alpha_1\cdots\alpha_n} &= \frac{2n}{n+2}\nabla_{(\alpha_1}{}^{\gamma}\lambda^n_{\alpha_2\cdots\alpha_n)\gamma} + \frac{n}{16}\psi_1\lambda^n_{\alpha_1\cdots\alpha_n},\\
\delta\chi^{n,-}_{\alpha_1\cdots\alpha_{n-2}} &= \frac{n-1}{n+1}\nabla^{\gamma\delta}\lambda^n_{\gamma\delta\alpha_1\cdots\alpha_{n-2}}.
\end{aligned} \tag{2.142}$$

Expanding λ^n as $\lambda^n = \lambda^n_{even} + \psi_1\lambda^n_{odd}$, we can use λ^n_{even} to set $\chi^{n,0}_{odd} = 0$, and $\chi^{n,+}_{odd}, \chi^{n,-}_{odd}$ transform under gauge transformation generated by the residual gauge parameter λ^n_{odd} as

$$\begin{aligned}
\delta\chi^{n,+}_{odd}(y) &= -\nabla^{+}\lambda_{odd}(y),\\
\delta\chi^{n,-}_{odd}(y) &= -\frac{1}{n(n+1)}\nabla^{-}\lambda_{odd}(y).
\end{aligned} \tag{2.143}$$

It is very useful to rewrite the equations of motion in the metric-like formulation. In the metric like formulation, we have the metric like field $\Phi_{\mu_1\cdots\mu_s}$ which is totally symmetric and satisfies the double traceless condition:

$$\Phi^{\mu\nu}{}_{\mu\nu\mu_5\cdots\mu_s} = 0. \tag{2.144}$$

$\Phi_{\mu_1\cdots\mu_s}$ satisfies the Fronsdal equation (2.39), and transforms under the gauge transformation as (2.40).

We show that the Fronsdal equation (2.39) and the frame-like equation (2.139) are equivalent. Let us decompose $\Phi_{\mu_1\cdots\mu_s}$ into the irreducible representation of the Lorentz group as in (2.37). Plugging this in to (2.39), we obtain

$$
\begin{aligned}
&(\Box - m^2)\xi_{\mu_1\cdots\mu_s} + (\Box - m^2) g_{(\underline{\mu_1\mu_2}}\chi_{\underline{\mu_3\cdots\mu_s})} - s\nabla_{(\underline{\mu_1}}\nabla^{\mu}\xi_{\mu\underline{\mu_2\cdots\mu_s})} \\
&+ (2s-3)\nabla_{(\underline{\mu_1}}\nabla_{\underline{\mu_2}}\chi_{\underline{\mu_3\cdots\mu_s})} - (s-2) g_{(\underline{\mu_1\mu_2}}\nabla_{\underline{\mu_3}}\nabla^{\mu}\chi_{\mu\underline{\mu_4\cdots\mu_s})} \\
&- 2(2s-1) g_{(\underline{\mu_1\mu_2}}\chi_{\underline{\mu_3\cdots\mu_s})} = 0.
\end{aligned}
\tag{2.145}
$$

Contracting this with $g^{\mu_1\mu_2}$, we get

$$
\begin{aligned}
&(2s-1)(\Box - m^2)\chi_{\mu_3\cdots\mu_s} - s(s-1)\nabla^{\mu}\nabla^{\nu}\xi_{\mu\nu\mu_3\cdots\mu_s} + (2s-3)\Box\chi_{\mu_3\cdots\mu_s} \\
&+ (2s-3)(s-2)\nabla^{\mu}\nabla_{(\underline{\mu_3}}\chi_{\mu\underline{\mu_4\cdots\mu_s})} - 2(s-2)\nabla_{(\underline{\mu_3}}\nabla^{\mu}\chi_{\mu\underline{\mu_4\cdots\mu_s})} \\
&- (s-2)(s-3) g_{(\underline{\mu_3\mu_4}}\nabla^{\mu}\nabla^{\nu}\chi_{\mu\nu\underline{\mu_5\cdots\mu_s})} - 2(2s-1)^2\chi_{\mu_3\cdots\mu_s} = 0.
\end{aligned}
\tag{2.146}
$$

By using the formula

$$
\nabla^{\mu}\nabla_{(\underline{\mu_3}}\chi_{\mu\underline{\mu_4\cdots\mu_s})} = \nabla_{(\underline{\mu_3}}\nabla^{\mu}\chi_{\mu\underline{\mu_4\cdots\mu_s})} - (s-1)\chi_{\mu_3\cdots\mu_s},
\tag{2.147}
$$

we can simplify (2.146) as

$$
\begin{aligned}
&(2s-1)(\Box - m^2)\chi_{\mu_3\cdots\mu_s} - s(s-1)\nabla^{\mu}\nabla^{\nu}\xi_{\mu\nu\mu_3\cdots\mu_s} + (d+2s-5)\Box\chi_{\mu_3\cdots\mu_s} \\
&+ (2s-5)(s-2)\nabla_{(\underline{\mu_3}}\nabla^{\mu}\chi_{\mu\underline{\mu_4\cdots\mu_s})} - (2s-3)(s-2)(s-1)\chi_{\mu_3\cdots\mu_s} \\
&- 2(2s-1)^2\chi_{\mu_3\cdots\mu_s} - (s-2)(s-3) g_{(\underline{\mu_3\mu_4}}\nabla^{\mu}\nabla^{\nu}\chi_{\mu\nu\underline{\mu_5\cdots\mu_s})} = 0.
\end{aligned}
\tag{2.148}
$$

Defining

$$
\begin{aligned}
\xi^s(y) &= y^{\alpha_1}\cdots y^{\alpha_{2s}}(e_0^{\mu_1})_{\alpha_1\alpha_2}\cdots(e_0^{\mu_s})_{\alpha_{2s-1}\alpha_{2s}}\xi_{\mu_1\cdots\mu_s}, \\
\chi^s(y) &= y^{\alpha_1}\cdots y^{\alpha_{2s}}(e_0^{\mu_1})_{\alpha_1\alpha_2}\cdots(e_0^{\mu_{s-2}})_{\alpha_{2s-5}\alpha_{2s-4}}\chi_{\mu_1\cdots\mu_{s-2}},
\end{aligned}
\tag{2.149}
$$

we can write (2.145) and (2.148) as

$$
\begin{aligned}
&\Box_{AdS}\xi^s - s(s-3)\xi^s + \frac{16}{2s-1}\nabla^+\nabla^-\xi^s + (2s-3)(\nabla^+)^2\chi^s = 0, \\
&\Box_{AdS}\chi^s - (s^2-s+1)\chi^s - \frac{4}{s-1}\nabla^+\nabla^-\chi^s - \frac{64}{(2s-1)(s-1)(2s-3)}(\nabla^-)^2\xi^s = 0.
\end{aligned}
\tag{2.150}
$$

We can then identify (2.140) and (2.150) by

$$\chi_{odd}^{2s-2,+} = \xi^s, \qquad \chi_{odd}^{2s-2,-} = -\frac{2s-3}{32(s-1)}\chi^s. \tag{2.151}$$

Later, we will also write $\chi_{odd}^{2s-2,\pm}$ as $\chi_{odd}^{(s),\pm}$ for convenience.

Let us also analyze the gauge transformation. Plugging (2.37) into (2.40), we have

$$\delta\xi_{\mu_1\cdots\mu_s} + g_{(\underline{\mu_1\mu_2}}\delta\chi_{\underline{\mu_3\cdots\mu_s})} = \nabla_{(\mu_1}\eta_{\mu_2\cdots\mu_s)}. \tag{2.152}$$

Contracting this with $g^{\mu_1\mu_2}$, we obtain

$$\delta\chi_{\mu_3\cdots\mu_s} = \frac{s-1}{2s-1}\nabla^\mu\eta_{\mu\mu_3\cdots\mu_s}. \tag{2.153}$$

It follows that

$$\begin{aligned} \delta\xi^s(y) &= \nabla^+\eta^s(y), \\ \delta\chi^s(y) &= -\frac{16}{(2s-1)(2s-3)}\nabla^-\eta^s(y). \end{aligned} \tag{2.154}$$

The gauge transformations (2.143) and (2.154) are also equivalent by the identification (2.151).

2.A.3 Derivation of higher spin boundary-to-bulk propagator in modified de Donder gauge

The Fronsdal equation (2.39) can be easily solved in the modified de Donder gauge proposed by Metsaev in [28]. As in (2.28), we define the generating function $\Phi^s(x|Y)$ of the metric-like higher spin gauge field $\Phi^s_{\mu_1\cdots\mu_s}$. The field $\Phi^s(x|Y)$ is related to $\chi^{2s-2,+}$ and $\chi^{2s-2,+}$ by

$$\begin{aligned} \chi_{odd}^{2s-2,+}(y) &= \xi^s(y) = \Phi^s(Y)\big|_{Y^A\to e^A{}_{\alpha\beta}y^\alpha y^\beta}, \\ \chi_{odd}^{2s-2,-}(y) &= -\frac{2s-3}{32(s-1)}\chi^s(y) = -\frac{2s-3}{64(2s-1)(s-1)}\frac{\partial^2\Phi^s(Y)}{\partial Y^2}\big|_{Y^A\to e^A{}_{\alpha\beta}y^\alpha y^\beta}. \end{aligned} \tag{2.155}$$

Using the variable Y^A, we can rewrite the Fronsdal equation (2.39), the gauge transformation (2.40), and the double traceless condition (2.144) as

$$\begin{aligned}
&\Big(\Box_{AdS} - s(s-3) - Y^A D^A \frac{\partial}{\partial Y^B} D^B \\
&\qquad + \frac{1}{2} Y^A D^A Y^B D^B \frac{\partial}{\partial Y^C}\frac{\partial}{\partial Y^C} - Y^A Y^A \frac{\partial}{\partial Y^B}\frac{\partial}{\partial Y^B}\Big) \Phi^s(x|Y) = 0, \\
&\delta \Phi^s(x|Y) = Y^A D^A \eta^s(x|Y), \\
&\left(\frac{\partial^2}{\partial Y^2}\right)^2 \Phi^s(x|Y) = 0,
\end{aligned} \tag{2.156}$$

where D^A is the covariant derivative acting both on explicit frame indices as well as on indices contracted with Y^A; in particular $\Box_{AdS} = D^A D_A$. As proposed by Metsaev [28], one then perform a linear transformation:

$$\phi(x|Y) = z^{-\frac{1}{2}} \mathcal{N} \Pi^{\phi\Phi} \Phi^s(x|Y), \tag{2.157}$$

and the inverse of it is

$$\Phi^s(x|Y) = z^{\frac{1}{2}} \Pi^{\Phi\phi} \mathcal{N} \phi(x|Y), \tag{2.158}$$

where the various operators are defined as

$$\begin{aligned}
&\mathcal{N} \equiv \left(\frac{2^{N_z}\Gamma(N_{\vec{Y}} + N_z - \frac{1}{2})\Gamma(2N_{\vec{Y}} - 1)}{\Gamma(N_{\vec{Y}} - \frac{1}{2})\Gamma(2N_{\vec{Y}} + N_z - 1)}\right)^{1/2}, \\
&\Pi^{\phi\Phi} \equiv \Pi_{\vec{Y}} + \vec{Y}^2 \frac{1}{4(N_{\vec{Y}}+1)} \Pi_{\vec{Y}} \left(\frac{\partial^2}{\partial \vec{Y}^2} + \frac{N_{\vec{Y}}+1}{N_{\vec{Y}}} \frac{\partial^2}{\partial Y^{z2}}\right), \\
&\Pi^{\Phi\phi} \equiv \Pi_Y + Y^2 \frac{1}{2(2N_Y+3)} \Pi_Y \left(\frac{\partial^2}{\partial \vec{Y}^2} - \frac{2}{2N_Y+1} \frac{\partial^2}{\partial Y^{z2}}\right), \\
&\Pi_{\vec{Y}} \equiv \Pi(\vec{Y}, 0, N_{\vec{Y}}, \frac{\partial}{\partial \vec{Y}}, 0, 2), \qquad \Pi_Y \equiv \Pi(\vec{Y}, Y^z, N_Y, \frac{\partial}{\partial \vec{Y}}, \frac{\partial}{\partial Y^z}, 3), \\
&\Pi(\vec{Y}, Y^z, A, \frac{\partial}{\partial \vec{Y}}, \frac{\partial}{\partial Y^z}, B) \equiv \sum_{n=0}^{\infty} (Y^2)^n \frac{(-)^n \Gamma(A + \frac{B-2}{2} + n)}{4^n n! \Gamma(A + \frac{B-2}{2} + 2n)} \left(\frac{\partial^2}{\partial Y^2}\right)^n, \\
&N_{\vec{Y}} = \vec{Y} \cdot \frac{\partial}{\partial \vec{Y}}, \qquad N_z = Y^z \frac{\partial}{\partial Y^z}, \qquad N_Y \equiv N_{\vec{Y}} + N_z.
\end{aligned} \tag{2.159}$$

The modified de Donder gauge condition written in terms of the field $\phi(x|Y)$ is:

$$\bar{C}\phi(x|Y) = 0, \tag{2.160}$$

where

$$\begin{aligned}
&\bar{C} \equiv \frac{\partial}{\partial \vec{Y}} \cdot \vec{\partial} - \frac{1}{2}\vec{Y} \cdot \vec{\partial} \frac{\partial^2}{\partial \vec{Y}^2} + \frac{1}{2} e_1 \frac{\partial^2}{\partial \vec{Y}^2} - \bar{e}_1 \Pi', \\
&\Pi' \equiv 1 - \vec{Y}^2 \frac{1}{4(N_{\vec{Y}} + 1)} \frac{\partial^2}{\partial \vec{Y}^2}, \\
&e_1 = e_{1,1}\left(\partial_z + \frac{2s - 3 - 2N_z}{2z}\right), \\
&\bar{e}_1 = \left(\partial_z - \frac{2s - 3 - 2N_z}{2z}\right)\bar{e}_{1,1}, \\
&e_{1,1} = Y^z f, \qquad \bar{e}_{1,1} = f \frac{\partial}{\partial Y^z}, \\
&f \equiv \varepsilon \left(\frac{2s - 2 - N_z}{2s - 2 - 2N_z}\right)^{1/2}, \qquad \varepsilon = \pm 1.
\end{aligned} \tag{2.161}$$

In this gauge, the equations of motion is simplified as

$$\left(\Box + \partial_z^2 - \frac{1}{z^2}(r - \frac{1}{2})(r - \frac{3}{2})\right)\phi_r = 0, \tag{2.162}$$

where $\phi_r(x|\vec{Y})$ are the components of $\phi(x|Y)$ expanded in Y^z as in (2.46), and the general solution of this equation is

$$\phi_r(\vec{p}, z|\vec{Y}) = C_1^r(\vec{p}, \vec{Y})\sqrt{z} J_{r-1}(z|\vec{p}|) + C_2^r(\vec{p}, \vec{Y})\sqrt{z} Y_{r-1}(z|\vec{p}|), \tag{2.163}$$

where we Fourier transformed $\phi_r(x|\vec{Y})$ as

$$\phi_r(x|\vec{Y}) = \int d^2x \ \phi_r(\vec{p}, z|\vec{Y}) \ e^{\vec{p}\cdot\vec{x}}. \tag{2.164}$$

Notice that $\vec{p}$ is imaginary momentum. We can Wick rotate back to the real momentum by $\vec{p} \to i\vec{p}$. For the purpose of computing the boundary-to-bulk propagator, we can simply replace $J_{r-1}(z|\vec{p}|)$ and $Y_{r-1}(z|\vec{p}|)$ by $i^{-r+1}K_{r-1}(x)$.

Next, let us solve for the functions $C_1^r(\vec{p},\vec{Y})$ and $C_2^r(\vec{p},\vec{Y})$ using the double traceless condition and the gauge condition. Let us first look at the reduced double traceless condition. It is convenient to define

$$Y^+ = Y^1 + iY^2 \quad \text{and} \quad Y^- = Y^1 - iY^2. \tag{2.165}$$

The double traceless condition (2.43) can be written as

$$\left(\frac{\partial}{\partial Y^+}\frac{\partial}{\partial Y^-}\right)^2 C^r(\vec{p},\vec{Y}) = 0. \tag{2.166}$$

The general solution of it is

$$C^r(\vec{p},\vec{Y}) = c^r_{++}(\vec{p})(Y^+)^r + c^r_{-+}(\vec{p})Y^-(Y^+)^{r-1} + c^r_{+-}(\vec{p})Y^+(Y^-)^{r-1} + c^r_{--}(\vec{p})(Y^-)^r. \tag{2.167}$$

for $r > 2$. For the $r = 1, 2$, we have

$$C^1(\vec{p},\vec{Y}) = c^1_+ Y^+ + c^1_- Y^- \quad \text{and} \quad C^2(\vec{Y}) = c^2_{++}(Y^+)^2 + c^2_{+-}Y^+Y^- + c^2_{--}(Y^-)^2. \tag{2.168}$$

Next, let us consider the gauge condition (2.160).

$$\begin{aligned}
\bar{C}\phi(x|Y) &= \left(\frac{\partial}{\partial\vec{Y}}\cdot\vec{p} - \frac{1}{2}\vec{Y}\cdot\vec{p}\frac{\partial^2}{\partial\vec{Y}^2} + \frac{1}{2}e_1\frac{\partial^2}{\partial\vec{Y}^2} - \bar{e}_1\Pi'\right)\sum_{r=0}^{s}(Y^z)^{s-r}\,\phi_r(\vec{p},z|\vec{Y}) \\
&= \Bigg[\frac{\partial}{\partial\vec{Y}}\cdot\vec{p} - \frac{1}{2}\vec{Y}\cdot\vec{p}\frac{\partial^2}{\partial\vec{Y}^2} + \frac{1}{2}Y^z\varepsilon\Big(\frac{2s+d-4-N_z}{2s+d-4-2N_z}\Big)^{1/2}\left(\partial_z + \frac{2s+d-5-2N_z}{2z}\right)\frac{\partial^2}{\partial\vec{Y}^2} \\
&\quad -\Big(\partial_z - \frac{2s+d-5-2N_z}{2z}\Big)\varepsilon\Big(\frac{2s+d-4-N_z}{2s+d-4-2N_z}\Big)^{1/2}\frac{\partial}{\partial Y^z}\Pi'\Bigg]\sum_{r=0}^{s}(Y^z)^{s-r}\,\phi_r(\vec{p},z|\vec{Y}) \\
&= \sum_{r=0}^{s}(Y^z)^{s-r}\Bigg[\frac{\partial}{\partial\vec{Y}}\cdot\vec{p} - \frac{1}{2}\vec{Y}\cdot\vec{p}\frac{\partial^2}{\partial\vec{Y}^2} + \frac{1}{2}Y^z\varepsilon\Big(\frac{s+r+d-4}{2r+d-4}\Big)^{1/2}\left(\partial_z + \frac{2r+d-5}{2z}\right)\frac{\partial^2}{\partial\vec{Y}^2} \\
&\quad -\varepsilon\Big(\partial_z - \frac{2r+d-3}{2z}\Big)\Big(\frac{s+r+d-3}{2r+d-2}\Big)^{1/2}\frac{s-r}{Y^z}\Pi'\Bigg]\phi_r(\vec{p},z|\vec{Y}) \\
&= \sum_{r=0}^{s}(Y^z)^{s-r}\Bigg[\frac{\partial}{\partial\vec{Y}}\cdot\vec{p} - \frac{1}{2}\vec{Y}\cdot\vec{p}\frac{\partial^2}{\partial\vec{Y}^2} + \frac{1}{2}Y^z\Big(\frac{s+r-2}{2r-2}\Big)^{1/2}\left(\partial_z + \frac{2r-3}{2z}\right)\frac{\partial^2}{\partial\vec{Y}^2} \\
&\quad -\varepsilon\Big(\partial_z - \frac{2r-1}{2z}\Big)\Big(\frac{s+r-1}{2r}\Big)^{1/2}\frac{s-r}{Y^z}\Pi'\Bigg]\phi_r(\vec{p},z|\vec{Y}).
\end{aligned} \tag{2.169}$$

The gauge condition can be written as

$$\left(\frac{\vec{p}}{p}\cdot\frac{\partial}{\partial\vec{Y}}-\frac{1}{2}\frac{\vec{p}}{p}\cdot\vec{Y}\frac{\partial^2}{\partial\vec{Y}^2}\right)\phi_{r+1}+\frac{1}{2}\Big(\frac{s+r}{2r+2}\Big)^{1/2}\left(\partial_z+\frac{2r+1}{2z}\right)\frac{\partial^2}{\partial\vec{Y}^2}\phi_{r+2}$$
$$-\,\varepsilon\Big(\partial_z-\frac{2r-1}{2z}\Big)\Big(\frac{s+r-1}{2r}\Big)^{1/2}(s-r)\Pi'\phi_r=0. \tag{2.170}$$

with $p\equiv|\vec{p}|$. Plugging (2.163) into (2.170), we obtain

$$\left(\frac{\vec{p}}{p}\cdot\frac{\partial}{\partial\vec{Y}}-\frac{1}{2}\frac{\vec{p}}{p}\cdot\vec{Y}\frac{\partial^2}{\partial\vec{Y}^2}\right)C^{r+1}+\frac{1}{2}\Big(\frac{s+r}{2r+2}\Big)^{1/2}\frac{\partial^2}{\partial\vec{Y}^2}C^{r+2}$$
$$+\,\varepsilon\Big(\frac{s+r-1}{2r}\Big)^{1/2}(s-r)\left(1-\vec{Y}^2\frac{1}{4(r-1)}\frac{\partial^2}{\partial\vec{Y}^2}\right)C^r=0, \tag{2.171}$$

or more explicitly,

$$\left[\frac{p^+}{p}\partial_++\frac{p^-}{p}\partial_--(\frac{p^+}{p}Y^-+\frac{p^-}{p}Y^+)\partial_+\partial_-\right]C^{r+1}+2\Big(\frac{s+r}{2r+2}\Big)^{1/2}\partial_+\partial_-C^{r+2}$$
$$+\,\varepsilon\Big(\frac{s+r-1}{2r}\Big)^{1/2}(s-r)\left(1-\vec{Y}^2\frac{1}{r-1}\partial_+\partial_-\right)C^r=0, \tag{2.172}$$

with $\partial_\pm=\partial_{Y^\pm}$. Plugging (2.167) and (2.168) into the above equation, we obtain

$$r\frac{p^+}{p}c^r_{++}(\vec{p})+\varepsilon\Big(\frac{s+r-2}{2(r-1)}\Big)^{1/2}(s-r+1)c^{r-1}_{++}(\vec{p})+(2-r)\frac{p^-}{p}c^r_{-+}(\vec{p})+2\Big(\frac{s+r-1}{2r}\Big)^{1/2}rc^{r+1}_{-+}(\vec{p})=0, \tag{2.173}$$

and

$$r\frac{p^-}{p}c^r_{--}(\vec{p})+\varepsilon\Big(\frac{s+r-2}{2(r-1)}\Big)^{1/2}(s-r+1)c^{r-1}_{--}(\vec{p})+(2-r)\frac{p^+}{p}c^r_{+-}(\vec{p})+2\Big(\frac{s+r-1}{2r}\Big)^{1/2}(r)c^{r+1}_{+-}(\vec{p})=0, \tag{2.174}$$

for $r>2$, and in the cases $r=1,2$,

$$2\frac{p^+}{p}c^2_{++}(\vec{p})+\varepsilon\Big(\frac{s}{2}\Big)^{1/2}(s-1)c^1_{+}(\vec{p})+2\Big(\frac{s+1}{4}\Big)^{1/2}2c^3_{-+}(\vec{p})=0,$$
$$2\frac{p^-}{p}c^2_{--}(\vec{p})+\varepsilon\Big(\frac{s}{2}\Big)^{1/2}(s-1)c^1_{-}(\vec{p})+2\Big(\frac{s+1}{4}\Big)^{1/2}2c^3_{+-}(\vec{p})=0, \tag{2.175}$$
$$\frac{p^+}{p}c^1_{+}(\vec{p})+\frac{p^-}{p}c^1_{-}(\vec{p})+2\Big(\frac{s}{2}\Big)^{1/2}c^2_{+-}(\vec{p})=0.$$

We can consistently set $c^r_{+-} = 0 = c^r_{-+}$ for $r > 2$, and obtain

$$r\frac{p^+}{p}c^r_{++}(\vec{p}) + \varepsilon\Big(\frac{s+r-2}{2(r-1)}\Big)^{1/2}(s-r+1)c^{r-1}_{++}(\vec{p}) + (2-r)\frac{p^-}{p}c^r_{-+}(\vec{p}) = 0, \tag{2.176}$$

and

$$r\frac{p^-}{p}c^r_{--}(\vec{p}) + \varepsilon\Big(\frac{s+r-2}{2(r-1)}\Big)^{1/2}(s-r+1)c^{r-1}_{--}(\vec{p}) + (2-r)\frac{p^+}{p}c^r_{+-}(\vec{p}) = 0, \tag{2.177}$$

for $r > 2$, and

$$\begin{aligned}
&2\frac{p^+}{p}c^2_{++}(\vec{p}) + \varepsilon\Big(\frac{s}{2}\Big)^{1/2}(s-1)c^1_{+}(\vec{p}) = 0,\\
&2\frac{p^-}{p}c^2_{--}(\vec{p}) + \varepsilon\Big(\frac{s}{2}\Big)^{1/2}(s-1)c^1_{-}(\vec{p}) = 0,\\
&\frac{p^+}{p}c^1_{+}(\vec{p}) + \frac{p^-}{p}c^1_{-}(\vec{p}) + 2\Big(\frac{s}{2}\Big)^{1/2}c^2_{+-}(\vec{p}) = 0,
\end{aligned} \tag{2.178}$$

for $r = 1, 2$. The solution to the above recursive equations is given by

$$\begin{aligned}
c^r_{++} &= \frac{s!}{(s-r)!r!}\sqrt{\frac{2^{s-r}(s-1)!(s+r-2)!}{(r-1)!(2s-2)!}}(-\varepsilon\frac{p^+}{p})^{s-r}c^s_{++},\\
c^r_{--} &= \frac{s!}{(s-r)!r!}\sqrt{\frac{2^{s-r}(s-1)!(s+r-2)!}{(r-1)!(2s-2)!}}(-\varepsilon\frac{p^-}{p})^{s-r}c^s_{--},
\end{aligned} \tag{2.179}$$

and

$$c^2_{+-}(\vec{p}) = \sqrt{\frac{2^{s-2}s!(s-1)!}{(2s-2)!}}(-\varepsilon\frac{p^+}{p})^s c^s_{++} + \sqrt{\frac{2^{s-2}s!(s-1)!}{(2s-2)!}}(-\varepsilon\frac{p^-}{p})^s c^s_{--}. \tag{2.180}$$

Starting from here and in what follows, we set $\varepsilon = -1$ and only consider the positively polarized fields by setting $c^s_{--} = 0$. Plugging (2.179) and (2.180) back to (2.167) and (2.168), then back to (2.163), and Wick rotating to the real momenta, we obtain

$$\begin{aligned}
&\phi(\vec{p}, z|\vec{Y}, Y^z)\\
&= \sum_{r=1}^{s} i^{1-r}\frac{s!}{(s-r)!r!}\sqrt{\frac{2^{s-r}(s-1)!(s+r-2)!}{(r-1)!(2s-2)!}}\left(\frac{p^+}{p}\right)^{s-r}(Y^z)^{s-r}(Y^+)^r c^s_{++}\sqrt{z}K_{r-1}(pz)\\
&\quad + i^{-1}\sqrt{\frac{2^{s-2}s!(s-1)!}{(2s-2)!}}\left(\frac{p^+}{p}\right)^{s} c^s_{++}Y^+Y^-(Y^z)^{s-2}\sqrt{z}K_1(pz).
\end{aligned} \tag{2.181}$$

Using the transformation (2.158), we arrive at the expression for the boundary to bulk propagator in momentum space, in the modified de Donder gauge,

$$\begin{aligned}
&\Phi^s(\vec{p}, z|Y) \\
&= z^{\frac{1}{2}}\Pi^{\Phi\phi}\mathcal{N}\phi(\vec{p}, z|\vec{Y}, Y^z) \\
&= \sum_{r=1}^{s}\sum_{n=0}^{\infty}\frac{(-1)^n i^{1-r}\Gamma(s-n-\frac{1}{2})}{4^n n!\Gamma(s-\frac{1}{2})}\frac{s!}{(s-r-2n)!r!}\left(\frac{p^+}{p}\right)^{s-r}Y^{2n}(Y^z)^{s-r-2n}(Y^+)^r c^s_{++} z K_{r-1}(pz) \\
&\quad+\sum_{n=0}^{\infty}\frac{(-1)^n i^{-1}\Gamma(s-n-\frac{1}{2})}{4^n n!\Gamma(s-\frac{1}{2})}\frac{(s-2)!}{(s-2-2n)!}\left(\frac{p^+}{p}\right)^{s}c^s_{++}Y^{2n}(Y^z)^{s-2-2n}Y^+Y^- zK_1(pz).
\end{aligned} \tag{2.182}$$

In terms of the frame-like fields, using (2.155), we have

$$\begin{aligned}
\chi^{(s),+}_{odd}(\vec{p}, z|y) =& c^s_{++}\sum_{r=0}^{s} i^r \frac{s!}{(s-r)!r!}p^{r-1}(p^+)^{s-r}(y^1)^{s+r}(y^2)^{s-r} zK_{r-1}(z|\vec{p}|), \\
\chi^{(s),-}_{odd}(\vec{p}, z|y) =& c^s_{++}\frac{s}{2(2s-1)}\sum_{r=0}^{s} i^r \frac{(s-2)!}{(s-r-2)!r!}p^{r-1}(p^+)^{s-r}(y^1)^{s+r-2}(y^2)^{s-r-2} zK_{r-1}(z|\vec{p}|).
\end{aligned} \tag{2.183}$$

2.B Second order in perturbation theory

2.B.1 A star-product relation

Let us write the following useful formula for the star-product:

$$A(y) * B(y) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{p=0}^{\infty}\frac{(m+p)!(n-m+p)!}{p!m!(n-m)!}A_{\alpha_1\cdots\alpha_p(\underline{\beta_1\cdots\beta_m}}B^{\alpha_1\cdots\alpha_p}{}_{\underline{\beta_{m+1}\cdots\beta_n})}\right)y^{\beta_1}\cdots y^{\beta_n} \tag{2.184}$$

where $A(y)$ and $B(y)$ have the expansions:

$$A(y) = \sum_{n=0}^{\infty} A_{\alpha_1\cdots\alpha_n}y^{\alpha_1}\cdots y^{\alpha_n}, \quad \text{and} \quad B(y) = \sum_{n=0}^{\infty} B_{\alpha_1\cdots\alpha_n}y^{\alpha_1}\cdots y^{\alpha_n}. \tag{2.185}$$

(2.184) follows from writing the (m-th) $*$ (n-th) term as

$$\begin{aligned}
&(A_{\alpha_1\cdots\alpha_m} y^{\alpha_1}\cdots y^{\alpha_m}) * \left(B_{\beta_1\cdots\beta_n} y^{\beta_1}\cdots y^{\beta_n}\right) \\
&= (-1)^m A^{\alpha_1\cdots\alpha_m}(y_{\alpha_1} + \frac{\partial}{\partial y^{\alpha_1}})\cdots(y_{\alpha_m} + \frac{\partial}{\partial y^{\alpha_m}}) B_{\beta_1\cdots\beta_n} y^{\beta_1}\cdots y^{\beta_n} \\
&= \sum_{p\le m,n} \frac{n!m!}{(m-p)!(n-p)!p!} A_{\alpha_1\cdots\alpha_p(\underline{\alpha_{p+1}\cdots\alpha_m}} B^{\alpha_1\cdots\alpha_p}{}_{\underline{\beta_{p+1}\cdots\beta_n})} y^{\alpha_{p+1}}\cdots y^{\alpha_m} y^{\beta_{p+1}}\cdots y^{\beta_n}.
\end{aligned} \tag{2.186}$$

2.B.2 Derivation of $U^{0,\mu}$ and $U^2_{\mu|\alpha\beta}$

The purpose of this subsection is to compute the RHS of (2.70).

By using the star-product relation (2.184), we obtain

$$\begin{aligned}
&[\Omega^{even}, C^{(1)}_{mat}]_* \\
&\quad = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{p=0}^{\infty} \frac{(m+p)!(x-m+p)!}{p!m!(n-m)!}(1-(-)^p)\Omega^{even}_{\alpha_1\cdots\alpha_p(\underline{\beta_1\cdots\beta_m}} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}{}_{\underline{\beta_{m+1}\cdots\beta_n})}\right) y^{\beta_1}\cdots y^{\beta_n}, \\
&\{\Omega^{odd}, C^{(1)}_{mat}\}_* \\
&\quad = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{p=0}^{\infty} \frac{(m+p)!(n-m+p)!}{p!m!(n-m)!}(1+(-)^p)\Omega^{odd}_{\alpha_1\cdots\alpha_p(\underline{\beta_1\cdots\beta_m}} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}{}_{\underline{\beta_{m+1}\cdots\beta_n})}\right) y^{\beta_1}\cdots y^{\beta_n}.
\end{aligned} \tag{2.187}$$

The U^0_μ and $U^2_{\mu|\alpha_1\alpha_2}$ are coefficients of the components in $-[\Omega^{even}, C^{(1)}_{mat}]_* + \psi_1\{\Omega^{odd}, C^{(1)}_{mat}\}_*$, which are independent and quadratic in y. They can be written as

$$U^{(0)}_\mu = \psi_1 \sum_{p=0}^{\infty} p!(1+(-)^p)\Omega^{odd}_{\mu|\alpha_1\cdots\alpha_p} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}, \tag{2.188}$$

and

$$\begin{aligned}
U^{(2)}_{\mu|\alpha\beta} &= -\sum_{p=0}^{\infty}(p+1)(p+1)!(1-(-)^p)\Omega^{even}_{\mu|\alpha_1\cdots\alpha_p(\alpha} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}{}_{\beta)} \\
&+ \psi_1\sum_{p=0}^{\infty}\frac{(p+2)!}{2}(1+(-)^p)\Omega^{odd}_{\mu|\alpha_1\cdots\alpha_p} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}{}_{\alpha\beta} + \psi_1\sum_{p=0}^{\infty}\frac{(p+2)!}{2}(1+(-)^p)\Omega^{odd}_{\mu|\alpha_1\cdots\alpha_p\alpha\beta} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}.
\end{aligned} \tag{2.189}$$

We first compute $\nabla^\mu U^{(0)}_\mu$:

$$\begin{aligned}
\nabla^\mu U^{(0)}_\mu =& -32\psi_1 \sum_{p=0}^{\infty} p!(1+(-)^p)\left(\nabla^{\alpha\beta}\Omega^{odd}_{\alpha\beta|\alpha_1\cdots\alpha_p} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p} + \Omega^{odd}_{\alpha\beta|\alpha_1\cdots\alpha_p}\nabla^{\alpha\beta} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}\right)\\
=& -32\psi_1 \sum_{p=0}^{\infty} p!(1+(-)^p)\Big(\nabla^{\alpha\beta}\chi^{p,+,odd}_{\alpha\beta\alpha_1\cdots\alpha_p} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p} + \nabla_{\alpha_1\alpha_2}\chi^{p,-,odd}_{\alpha_3\cdots\alpha_p} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}\\
&+\chi^{p,+,odd}_{\alpha\beta\alpha_1\cdots\alpha_p}\nabla^{\alpha\beta} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p} + \chi^{p,-,odd}_{\alpha_3\cdots\alpha_p}\nabla_{\alpha_1\alpha_2} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}\Big)\\
=&32\psi_1\sum_{p=0}^{\infty}(1+(-)^p)\Bigg[C^{(1),p}_{mat}(\partial_y)\left(\frac{\nabla^-\chi^{p,+}_{odd}(y)}{(p+2)(p+1)} + \nabla^+\chi^{p,-}_{odd}(y)\right)\\
&+\frac{(\nabla^+ C^{(1),p}_{mat})(\partial_y)\chi^{p,+}_{odd}(y)}{(p+2)(p+1)} + (\nabla^- C^{(1),p}_{mat})(\partial_y)\chi^{p,-}_{odd}(y)\Bigg],
\end{aligned} \tag{2.190}$$

where we have assumed the gauge condition $\chi^{p,0}_{odd} = 0$. Using (2.103) to express $\nabla^\pm C^{(1),p}_{mat}$ in terms of $C^{(1),p\pm2}_{mat}$, we have

$$\begin{aligned}
\nabla^\mu U^{(0)}_\mu =&32\psi_1\sum_{p=0}^{\infty}(1+(-)^p)\Bigg[C^{(1),p}_{mat}(\partial_y)\left(\frac{\nabla^-\chi^{p,+}_{odd}(y)}{(p+2)(p+1)} + \nabla^+\chi^{p,-}_{odd}(y)\right)\\
&+\psi_1\frac{C^{(1),p+2}_{mat}(\partial_y)\chi^{p,+}_{odd}(y)}{16} + \psi_1\frac{p(p+1)}{16}C^{(1),p-2}_{mat}(\partial_y)\chi^{p,-}_{odd}(y)\Bigg].
\end{aligned} \tag{2.191}$$

Next, we compute $(e^\mu_0)^{\alpha\beta}U^{(2)}_{\mu|\alpha\beta}$:

$$\begin{aligned}
(e^\mu_0)^{\alpha\beta}U^{(2)}_{\mu|\alpha\beta} =& \sum_{p=0}^{\infty}\frac{(p+3)(p+1)!}{2}(1-(-)^p)\chi^{p+1,0,even}_{\alpha_1\cdots\alpha_p\beta} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p\beta}\\
&+\psi_1\sum_{p=0}^{\infty}\frac{(p+2)!}{2}(1+(-)^p)\chi^{p+1,+,odd}_{\alpha_1\cdots\alpha_p\alpha\beta} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p\alpha\beta}\\
&+\psi_1\sum_{p=0}^{\infty}\frac{(p+3)(p+2)p!}{2}(1+(-)^p)\chi^{p,-,odd}_{\alpha_1\cdots\alpha_p} C^{(1)}_{mat}{}^{\alpha_1\cdots\alpha_p}\\
=&\sum_{p=0}^{\infty}\frac{(p+3)(1-(-)^p)}{2}C^{(1),p+1}_{mat}(\partial_y)\chi^{p+1,0}_{even}(y) + \psi_1\sum_{p=0}^{\infty}\frac{(1+(-)^p)}{2}C^{(1),p+2}_{mat}(\partial_y)\chi^{p,+}_{odd}(y)\\
&+\psi_1\sum_{p=0}^{\infty}\frac{(p+3)(p+2)(1+(-)^p)}{2}C^{(1),p}_{mat}(\partial_y)\chi^{p+2,-}_{odd}(y),
\end{aligned} \tag{2.192}$$

where we have assumed the gauge $\chi^{p,0}_{odd} = 0$. Using (2.138) to express $\chi^{p+1,0}_{even}$ in terms of $\chi^{p+1,+}_{odd}$ and $\chi^{p+1,-}_{odd}$, we have

$$(e_0^\mu)^{\alpha\beta} U^{(2)}_{\mu|\alpha\beta} = \sum_{p=0}^{\infty} (1-(-)^p) C^{(1),p+1}_{mat}(\partial_y) \left[\frac{4(p+1)}{(p+3)(p+2)} \nabla^- \chi^{p+1,+}_{odd}(y) - 4(p+3)\nabla^+ \chi^{p+1,-}_{odd}(y) \right]$$
$$+ \psi_1 \sum_{p=0}^{\infty} \frac{(1+(-)^p)}{2} C^{(1),p+2}_{mat}(\partial_y) \chi^{p,+}_{odd}(y) + \psi_1 \sum_{p=0}^{\infty} \frac{(p+3)(p+2)(1+(-)^p)}{2} C^{(1),p}_{mat}(\partial_y) \chi^{p+2,-}_{odd}(y), \tag{2.193}$$

Adding the two terms (2.191) and (2.193), we obtain

$$\begin{aligned} &\nabla^\mu U^{(0)}_\mu + 4\psi_1 (e_0^\mu)^{\alpha\beta} U^{(2)}_{\mu|\alpha\beta} \\ &= 4\sum_{p=0}^{\infty} (1+(-)^p) \left[C^{(1),p+2}_{mat}(\partial_y) \chi^{p,+}_{odd}(y) + (p+1)p C^{(1),p-2}_{mat}(\partial_y) \chi^{p,-}_{odd}(y) \right] \\ &\quad + 16\psi_1 \sum_{p=2}^{\infty} (1+(-)^p) C^{(1),p}_{mat}(\partial_y) \left[\frac{1}{(p+1)} \nabla^- \chi^{p,+}_{odd}(y) - p\nabla^+ \chi^{p,-}_{odd}(y) \right]. \end{aligned} \tag{2.194}$$

2.B.3 Computation of the three point function

In this subsection, we compute the three point function of a higher spin current with two scalars by explicitly evaluating the integral (2.75).

To begin with, let us turn on boundary sources only for the C_{even} component of the scalars in (2.75). It is convenient to decompose Ξ_s as $\Xi_s = \Xi_s^+ + \Xi_s^0 + \Xi_s^-$, with $\Xi_s^{\pm/0}$ being the homogeneous polynomials in y of degree $2s$, $2s-2$, and $2s-4$, respectively. The action (2.75) splits into three terms. The terms with $\Xi_s^\pm$ have already been of the form (2.73). For the term with Ξ_s^0, we need to perform an integration by part:

$$
\begin{aligned}
&\int dx^2 \left(\frac{dz}{z^3}\right) \Xi_s^0(\partial_y)\delta C_{mat}^{(1),0} C_{mat}^{(1),2s-2} \\
&= \int dx^2 \left(\frac{dz}{z^3}\right) 32\psi_1 \left(\frac{1}{(2s-1)}\nabla^- \chi_{odd}^{(s),+}(\partial_y) - (2s-2)\nabla^+ \chi_{odd}^{(s),-}(\partial_y)\right)\delta C_{mat}^{(1),0} C_{mat}^{(1),2s-2} \\
&= \int dx^2 \left(\frac{dz}{z^3}\right)\Big[-4\frac{1}{(2s-1)}\chi_{odd}^{(s),+}(\partial_y)\delta C_{mat}^{(1),2} C_{mat}^{(1),2s-2} - 4s\chi_{odd}^{(s),+}(\partial_y)\delta C_{mat}^{(1),0} C_{mat}^{(1),2s} \\
&\quad + 4(2s_{mat}-2)\chi_{odd}^{(s),-}(\partial_y)\delta C_{mat}^{(1),2}(\partial_y) C_{mat}^{(1),2s-2} + 2(2s-2)^2(2s-1)\chi_{odd}^{(s),-}(\partial_y)\delta C_{mat}^{(1),0} C_{mat}^{(1),2s-4}\Big],
\end{aligned}
\tag{2.195}
$$

where we have used (2.103) to express $\nabla^{\pm} C_{mat}^{(1),p}$ in terms of $C_{mat}^{(1),p\pm 2}$. The variation of the action δS is then given by

$$\begin{aligned}
\delta S = &\int d^2x \left(\frac{dz}{z^3}\right) \Bigg[\chi^{(s),+}_{odd}(\partial_y) \left((8-4s)\delta C^{(1),0}_{mat} C^{(1),2s}_{mat} - 4\frac{1}{(2s-1)}\delta C^{(1),2}_{mat} C^{(1),2s-2}_{mat}\right) \\
&+ 4\chi^{(s),-}_{odd}(\partial_y)\left((2s-2)\delta C^{(1),2}_{mat}(\partial_y) C^{(1),2s-2}_{mat} + 2(s-1)(s+1)(2s-1)\delta C^{()1,0}_{mat} C^{(1),2s-4}_{mat}\right)\Bigg] \\
= &-\int d^2x \left(\frac{dz}{z^3}\right)\Bigg[\nabla^+\lambda(\partial_y)\left((8-4s)\delta C^{(1),0}_{mat} C^{(1),2s}_{mat} - 4\frac{1}{(2s-1)}\delta C^{(1),2}_{mat} C^{(1),2s-2}_{mat}\right) \\
&- 4\nabla^-\lambda(\partial_y)\left(\frac{1}{(2s-1)}\delta C^{(1),2}_{mat}(\partial_y) C^{(1),2s-2}_{mat} + (s+1)\delta C^{(1),0}_{mat} C^{(1),2s-4}_{mat}\right)\Bigg] \\
= &-\int d^2x dz \partial_z \Bigg[\frac{1}{z^2}\lambda(\partial_y)\partial_{y^1}\partial_{y^2}\left((2-s)\delta C^{(1),0}_{mat} C^{(1),2s}_{mat} - \frac{1}{(2s-1)}\delta C^{(1),2}_{mat} C^{(1),2s-2}_{mat}\right) \\
&- \frac{1}{z^2}\left(\partial_{y^1}\partial_{y^2}\lambda\right)(\partial_y)\left(\frac{1}{2s-1}\delta C^{(1),2}_{mat}(\partial_y) C^{(1),2s-2}_{mat} + (s+1)\delta C^{(1),0}_{mat} C^{(1),2s-4}_{mat}\right)\Bigg] \\
= &\lim_{z\to 0}\int d^2x \frac{1}{z^2}\Bigg[\lambda(\partial_y)\partial_{y^1}\partial_{y^2}\left((2-s)\delta C^{(1),0}_{mat} C^{(1),2s}_{mat} - \frac{1}{(2s-1)}\delta C^{(1),2}_{mat} C^{(1),2s-2}_{mat}\right) \\
&+ \left(\partial_{y^1}\partial_{y^2}\lambda\right)(\partial_y)\left(\frac{1}{2s-1}\delta C^{(1),2}_{mat}(\partial_y) C^{(1),2s-2}_{mat} + (s+1)\delta C^{(1),0}_{mat} C^{(1),2s-4}_{mat}\right)\Bigg] \\
= &4\lim_{z\to 0}\int d^2x \sum_{r=1}^{2s-1}\frac{z^{r-s-2}}{(x^- - x_3^-)^r}\Bigg[(\partial_{y^2})^{2s-r}(-\partial_{y^1})^r\left((2-s)\delta C^{(1),0}_{mat} C^{(1),2s}_{mat} - \frac{1}{(2s-1)}\delta C^{(1),2}_{mat} C^{(1),2s-2}_{mat}\right) \\
&- (2s-r-1)(r-1)(\partial_{y^2})^{2s-r-2}(-\partial_{y^1})^{r-2}\left(\frac{1}{2s-1}\delta C^{(1),2}_{mat}(\partial_y) C^{(1),2s-2}_{mat} + (s+1)\delta C^{(1),0}_{mat} C^{(1),2s-4}_{mat}\right)\Bigg] \\
\equiv &\ \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4,
\end{aligned} \tag{2.196}$$

where we substituted the boundary to bulk propagator for $\chi^{(s),+}_{odd}$ and $\chi^{(s),-}_{odd}$ in the "pure gauge" form, and we also performed the similar step as illustrated in (2.74), and we used (2.103) again to express $\nabla^{\pm} C^{(1),p}_{mat}$ in terms of $C^{(1),p\pm 2}_{mat}$. For the convenience of the later computation, we have split δS into four terms $\delta S = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4$. We will compute these four terms one by one in the following. The next step is to substitute the boundary-to-bulk propagator for the master field $C^{(1)}_{mat}$. We first expand $C^{(1)}_{mat}$ as

$$
\begin{aligned}
C^{(1)}_{mat}(y) &= \left(1+\psi_1\frac{1+\tilde{\psi}_1}{2}y\Sigma y\right) e^{\frac{\psi_1}{2}y\Sigma y}K^{1+\frac{\tilde{\psi}_1}{2}} \\
&= \sum_{s=0}^{\infty}\frac{1}{s!}\left(1+s(1+\tilde{\psi}_1)\right)\left(\frac{\psi_1}{2}\right)^s (y\Sigma y)^s K^{1+\frac{\tilde{\psi}_1}{2}} \\
&= \sum_{s=0}^{\infty}\frac{\psi_1^s}{s!}\left(1+s(1+\tilde{\psi}_1)\right)\left[(z-\frac{x^+x^-}{z})y^1y^2-(y^1)^2x^-+(y^2)^2x^+\right]^s K^{1+\frac{\tilde{\psi}_1}{2}+s} \\
&= \sum_{s=0}^{\infty}\frac{\psi_1^s}{s!}\left(1+s(1+\tilde{\psi}_1)\right)\sum_{u=0}^{s}\sum_{w=0}^{u}\sum_{v=0}^{u-w}\frac{s!}{(s-u)!(u-w-v)!w!v!} \\
&\quad\times z^{u-w-2v}(-x^-)^{w+v}(x^+)^{s-u+v}(y^1)^{u+w}(y^2)^{2s-u-w}K^{1+\frac{\tilde{\psi}_1}{2}+s}.
\end{aligned}
\tag{2.197}
$$

In particular, the piece of homogeneous degree $2s$ is given by

$$
\begin{aligned}
C^{(1),2s}_{mat}(y) = \frac{\psi_1^s}{s!}\left(1+s(1+\tilde{\psi}_1)\right)\sum_{u=0}^{s}\sum_{w=0}^{u}\sum_{v=0}^{u-w}\frac{s!}{(s-u)!(u-w-v)!w!v!} \\
\times z^{u-w-2v}(-x^-)^{w+v}(x^+)^{s-u+v}(y^1)^{u+w}(y^2)^{2s-u-w}K^{1+\frac{\tilde{\psi}_1}{2}+s}.
\end{aligned}
\tag{2.198}
$$

where $K=\frac{z}{z^2+x^2}$ is the scalar boundary-to-bulk propagator. Near the boundary, $K^{1+\frac{\tilde{\psi}_1}{2}+s}$ has the following expansion

$$
K^{1+\frac{\tilde{\psi}_1}{2}+s}\to\pi\sum_{q=0}^{s}\frac{\Gamma(s-q+\frac{\tilde{\psi}_1}{2})}{q!\Gamma(1+s+\frac{\tilde{\psi}_1}{2})}z^{2q+1-\frac{\tilde{\psi}_1}{2}-s}(\partial_{x^+}\partial_{x^-})^q\delta^2(x)+z^{1+\frac{\tilde{\psi}_1}{2}+s}\frac{1}{x^{2+\tilde{\psi}_1+2s}}+\cdots,
\tag{2.199}
$$

where we keep only the leading analytic term and the first s contact terms. The subleading terms will not contribute to the three point function.

Let us first compute δS_1.

$$
\begin{aligned}
&\delta S_1 \\
&= 4\lim_{z\to 0}\int d^2x_0 \sum_{r=1}^{2s-1}(2-s)\frac{1}{(x_{03}^-)^r} z^{r-s-2}(\partial_{y^2})^{2s-r}(-\partial_{y^1})^r \delta C_{mat}^{(1),0}(x_{01})C_{mat}^{(1),2s}(x_{02}|y) \\
&= 4\lim_{z\to 0}\int d^2x_0 \sum_{r=1}^{2s-1}\psi_1^s\left(1+s(1+\tilde{\psi}_1)\right)\sum_{u=0}^{s}\sum_{v=0}^{2u-r}\frac{(2-s)r!(2s-r)!(-1)^{-u+v}}{(s-u)!(r-u)!(2u-r-v)!v!} \\
&\quad \times z^{2u-2v-s-2}(x_{02}^-)^{r-u+v}(x_{02}^+)^{s-u+v}\frac{1}{(x_{03}^-)^r}K_{01}^{1+\frac{\tilde{\psi}_1}{2}}K_{02}^{1+\frac{\tilde{\psi}_1}{2}+s} \\
&= 4\int d^2x_0 \sum_{r=1}^{2s-1}\psi_1^s\left(1+s(1+\tilde{\psi}_1)\right)\sum_{u=0}^{s}\sum_{v=0}^{2u-r}\frac{(2-s)r!(2s-r)!(-1)^{-u+v}}{(s-u)!(r-u)!(2u-r-v)!v!} \\
&\quad \times \Bigg[\pi^{\frac{3}{2}}\frac{\Gamma(\frac{1}{2}\tilde{\psi}_1)}{\Gamma(\frac{1}{2})\Gamma(1+\frac{\tilde{\psi}_1}{2})}\delta^2(x_{01})\frac{1}{x_{02}^{2+\tilde{\psi}_1+2s}}(x_{02}^-)^r(x_{02}^+)^s\delta_{u,v}\frac{1}{(x_{03}^-)^r} \\
&\quad + \delta_{v,u+q-s}\pi\sum_{q=0}^{s}\frac{\Gamma(s-q+\frac{\tilde{\psi}_1}{2})}{\Gamma(1+s+\frac{\tilde{\psi}_1}{2})}\delta^2(x_{02})\sum_{n=0}^{q}\frac{q!(q+r-s)!}{(q-n)!n!(r-s+n)!}(x_{02}^-)^{r-s+n}\partial_{x_0^-}^n\left(\frac{1}{(x_{03}^-)^r}\frac{1}{x_{01}^{2+\tilde{\psi}_1}}\right)\Bigg],
\end{aligned}
\tag{2.200}
$$

where we have substituted the boundary-to-bulk propagator for $\delta C_{mat}^{(1),0}(x_{01})$ and $C_{mat}^{(1),2s}(x_{02}|y)$, and the K_{ij} stands for $K\big|_{x\to x_{ij}}$, and we have substituted the expansion (2.199) for K_{ij}. Integrating out the delta functions gives

$$
\begin{aligned}
\delta S_1 = 4&\sum_{r=1}^{2s-1}(2-s)\psi_1^s\left(1+s(1+\tilde{\psi}_1)\right)\Bigg[2\pi\tilde{\psi}_1\frac{(2s-r)!}{(s-r)!}\frac{1}{x_{12}^{2+\tilde{\psi}_1}(x_{12}^-)^{s-r}(x_{13}^-)^r} \\
&+\sum_{u=0}^{s}\sum_{q=0}^{s}\frac{r!(2s-r)!\Gamma(s-q+\frac{\tilde{\psi}_1}{2})q!(-1)^{q-s}}{(s-u)!(r-u)!(u-r-q+s)!(u+q-s)!\Gamma(1+s+\frac{\tilde{\psi}_1}{2})(s-r)!}\pi\partial_{x_2^-}^{s-r}\left(\frac{1}{(x_{23}^-)^r x_{21}^{2+\tilde{\psi}_1}}\right)\Bigg].
\end{aligned}
\tag{2.201}
$$

Similarly, let us compute δS_2 and δS_3 as follows. Substituting the boundary-to-bulk propagator for the master field $C^{(1)}_{mat}$, we have

$$\delta S_2 = -4\lim_{z\to 0}\int d^2x_0 \sum_{r=1}^{2s-1}\frac{z^{r-s-2}}{(2s-1)}\frac{1}{(x_{03}^-)^r}(\partial_{y^2})^{2s-r}(-\partial_{y^1})^r\delta C^{(1),2}_{mat}(x_{01})C^{(1),2s-2}_{mat}(x_{02}|y)$$
$$= -4\lim_{z\to 0}\int d^2x_0 \sum_{r=1}^{2s-1}\frac{1}{(2s-1)}\frac{1}{(x_{03}^-)^r}\psi_1^s\left(1+(s-1)(1+\tilde{\psi}_1)\right)(2+\tilde{\psi}_1)K_{01}^{2+\frac{\tilde{\psi}_1}{2}}K_{02}^{\frac{\tilde{\psi}_1}{2}+s}$$
$$\times\Bigg[\sum_{u=0}^{s-1}\sum_{v=0}^{2u-r+1}\frac{r!(2s-r)!(-1)^r}{(s-u-1)!(2u-r+1-v)!(r-u-1)!v!}$$
$$\times\left(z-\frac{x_{01}^+x_{01}^-}{z}\right)z^{2u-2v-s-1}(-x_{02}^-)^{r-u+v-1}(x_{02}^+)^{s-u+v-1}$$
$$+\sum_{u=0}^{s-1}\sum_{v=0}^{2u-r+2}\frac{r!(2s-r)!(-1)^r}{(s-u-1)!(2u-r+2-v)!(r-u-2)!v!}(-x_{01}^-)z^{2u-2v-s}(-x_{02}^-)^{r-u+v-2}(x_{02}^+)^{s-u+v-1}$$
$$+\sum_{u=0}^{s-1}\sum_{v=0}^{2u-r}\frac{r!(2s-r)!(-1)^r}{(s-u-1)!(2u-r-v)!(r-u)!v!}(x_{01}^+)z^{2u-2v-s-2}(-x_{02}^-)^{r-u+v}(x_{02}^+)^{s-u+v-1}\Bigg], \tag{2.202}$$

and

$$\delta S_3 = -4\lim_{z\to 0}\int d^2x_0 \sum_{r=1}^{2s-1}\frac{z^{r-s-2}}{(2s-1)}\frac{1}{(x_{03}^-)^r}(2s-r-1)(r-1)$$
$$\times(\partial_{y^2})^{2s-r-2}(-\partial_{y^1})^{r-2}\delta C^{(1),2}_{mat}(x_{01}|\partial_y)C^{(1),2s-2}_{mat}(x_{02}|y)$$
$$= -4\lim_{z\to 0}\int d^2x_0 \sum_{r=1}^{2s-1}\frac{1}{(2s-1)}\frac{1}{(x_{03}^-)^r}(2s-r-1)(r-1)\psi_1^s\left(1+(s-1)(1+\tilde{\psi}_1)\right)(2+\tilde{\psi}_1)$$
$$\times K_{01}^{2+\frac{\tilde{\psi}_1}{2}}K_{02}^{\frac{\tilde{\psi}_1}{2}+s}\Bigg[\sum_{u=0}^{s-1}\sum_{v=0}^{2u-r+1}\frac{(r-1)!(2s-r-1)!(-1)^{r-1}}{(s-u-1)!(2u-r+1-v)!(r-u-1)!v!}$$
$$\times\left(z-\frac{x_{01}^+x_{01}^-}{z}\right)z^{2u-2v-s-1}(-x_{02}^-)^{r-u+v-1}(x_{02}^+)^{s-u+v-1}$$
$$+\sum_{u=0}^{s-1}\sum_{v=0}^{2u-r+2}\frac{(r-2)!(2s-r)!(-1)^{r-1}}{(s-u-1)!(2u-r+2-v)!(r-u-2)!v!}(x_{01}^-)z^{2u-2v-s}(-x_{02}^-)^{r-u+v-2}(x_{02}^+)^{s-u+v-1}$$
$$+\sum_{u=0}^{s-1}\sum_{v=0}^{2u-r}\frac{r!(2s-r-2)!}{(s-u-1)!(2u-r-v)!(r-u)!v!}(-1)^r(x_{01}^+)z^{2u-2v-s-2}(-x_{02}^-)^{r-u+v}(x_{02}^+)^{s-u-1+v}\Bigg]. \tag{2.203}$$

These two terms can be combined as

$$
\begin{aligned}
&\delta S_2 + \delta S_3 \\
&= -4 \lim_{z\to 0} \int d^2x_0 \sum_{r=1}^{2s-1} \psi_1^s \left(1+(s-1)(1+\tilde{\psi}_1)\right)(2+\tilde{\psi}_1) K_{01}^{2+\frac{\tilde{\psi}_1}{2}} K_{02}^{\frac{\tilde{\psi}_1}{2}+s} \frac{1}{(x_{03}^-)^r} \\
&\quad \times \Bigg[\sum_{u=0}^{s-1} \sum_{v=0}^{2u-r+1} \frac{(r-1)!(2s-r-1)!(-1)^r}{(s-u-1)!(2u-r+1-v)!(r-u-1)!v!} \\
&\qquad \times \left(z - \frac{x_1^+ x_1^-}{z}\right) z^{2u-2v-s-1} (-x_{02}^-)^{r-u+v-1} (x_{02}^+)^{s-u+v-1} \\
&\quad + \sum_{u=0}^{s-1} \sum_{v=0}^{2u-r+2} \frac{(r-1)!(2s-r)!(-1)^r}{(s-u-1)!(2u-r+2-v)!(r-u-2)!v!} (-x_{01}^-) z^{2u-2v-s} (-x_{02}^-)^{r-u+v-2} (x_{02}^+)^{s-u+v-1} \\
&\quad + \sum_{u=0}^{s-1} \sum_{v=0}^{2u-r} \frac{r!(2s-r-1)!(-1)^r}{(s-u-1)!(2u-r-v)!(r-u)!v!} (x_{01}^+) z^{2u-2v-s-2} (-x_{02}^-)^{r-u+v} (x_{02}^+)^{s-u+v-1} \Bigg] \\
&\equiv U_1 + U_2 + U_3,
\end{aligned}
\tag{2.204}
$$

where we have split $\delta S_2+\delta S_3$ into three terms U_1, U_2, U_3. These are computed as follows.

$$\begin{aligned}
U_1 = &-4\int d^2x_0 \sum_{r=1}^{2s-1} \psi_1^s \left(1+(s-1)(1+\tilde\psi_1)\right)(2+\tilde\psi_1) \\
&\times \sum_{u=0}^{s-1}\Bigg[-\frac{2\pi}{2+\tilde\psi_1}\delta^2(x_{01})\frac{1}{x_{02}^{\tilde\psi_1+2}}\frac{1}{(x_{02}^-)^{s-r}}\frac{1}{(x_{03}^-)^r}\frac{(r-1)!(2s-r-1)!}{(s-u-1)!(u-r+1)!(r-u-1)!u!} \\
&+\frac{4\pi}{2\tilde\psi_1+1}\delta^2(x_{01})\frac{1}{x_{02}^{\tilde\psi_1+2}}\frac{1}{(x_{02}^-)^{s-r}}\frac{1}{(x_{03}^-)^r}\frac{(r-1)!(2s-r-1)!}{(s-u-1)!(u-r+1)!(r-u-1)!u!} \\
&+\sum_{q=0}^{s-1}\frac{(r-1)!(2s-r-1)!\Gamma(s-1-q+\frac{\tilde\psi_1}{2})q!(-1)^{s+q+1}}{(s-u-1)!(u-r-q+s)!(r-u-1)!(q+u-s+1)!\Gamma(s+\frac{\tilde\psi_1}{2})(s-r)!} \\
&\qquad\times \pi\delta^2(x_{02})\partial_{x_0^-}^{s-r}\left(\frac{1}{x_{01}^{2+\tilde\psi_1}}\frac{1}{(x_{03}^-)^r}\right)\Bigg] \\
= &-4\sum_{r=1}^{2s-1}\psi_1^s\left(1+(s-1)(1+\tilde\psi_1)\right)(2+\tilde\psi_1)\Bigg[\frac{10\tilde\psi_1-8}{3}\pi\frac{(2s-r-1)!}{(s-r)!}\frac{1}{x_{12}^{\tilde\psi_1+2}(x_{12}^-)^{s-r}(x_{13}^-)^r} \\
&+\sum_{u=0}^{s-1}\sum_{q=0}^{s-1}\frac{(r-1)!(2s-r-1)!\Gamma(s-1-q+\frac{\tilde\psi_1}{2})q!(-1)^{s+q+1}}{(s-u-1)!(u-r-q+s)!(r-u-1)!(q+u-s+1)!\Gamma(s+\frac{\tilde\psi_1}{2})(s-r)!} \\
&\qquad\times\pi\partial_{x_2^-}^{s-r}\left(\frac{1}{x_{21}^{2+\tilde\psi_1}(x_{23}^-)^r}\right)\Bigg],
\end{aligned} \tag{2.205}$$

$$\begin{aligned}
U_2 = &-4\lim_{z\to0}\int d^2x_0\sum_{r=1}^{2s-1}\psi_1^s\left(1+(s-1)(1+\tilde\psi_1)\right)(2+\tilde\psi_1)\frac{1}{(x_{03}^-)^r} \\
&\times\sum_{u=0}^{s-1}\sum_{v=0}^{2u-r+2}\frac{(r-1)!(2s-r)!}{(s-u-1)!(2u-r+2-v)!(r-u-2)!v!}(-1)^r(-x_{01}^-)(-x_{02}^-)^{r-u+v-2}(x_{02}^+)^{s-u+v-1} \\
&\times\Bigg[\pi\sum_{q=0}^{1}\frac{\Gamma(1-q+\frac{\tilde\psi_1}{2})}{q!\Gamma(2+\frac{\tilde\psi_1}{2})}(\partial_{x_0^+}\partial_{x_0^-})^q\delta^2(x_{01})\frac{1}{x_{02}^{\tilde\psi_1+2s}}z^{2u-2v+2q} \\
&\frac{1}{x_{01}^{2+\tilde\psi_1+4}}\pi\sum_{q=0}^{s-1}\frac{\Gamma(s-1-q+\frac{\tilde\psi_1}{2})}{q!\Gamma(s+\frac{\tilde\psi_1}{2})}z^{2u-2v+2q+4-2s}(\partial_{x_0^+}\partial_{x_0^-})^q\delta^2(x_{02})\Bigg], \\
= &\ 0,
\end{aligned} \tag{2.206}$$

and

$$
\begin{aligned}
U_3 = & -4\sum_{r=1}^{2s-1}\psi_1^s\left(1+(s-1)(1+\tilde{\psi}_1)\right)(2+\tilde{\psi}_1)\left[\frac{4\pi}{1+2\tilde{\psi}_1}\frac{(2s-r-1)!}{(s-r-1)!}\partial_{x_1^-}\left(\frac{1}{x_{12}^{2+\tilde{\psi}_1}(x_{12}^-)^{s-r-1}(x_{13}^-)^r}\right)\right. \\
& +\sum_{q=0}^{s-1}\sum_{u=0}^{s-1}\frac{\Gamma(s-1-q+\frac{\tilde{\psi}_1}{2})r!(2s-r-1)!q!\pi(-1)^{1+s+q}}{\Gamma(s+\frac{\tilde{\psi}_1}{2})(s-u-1)!(u-r-q+s-1)!(r-u)!(q+1+u-s)!(s-r-1)!} \\
& \left.\times\partial_{x_2^-}^{s-r-1}\left(\frac{1}{x_{21}^{2+\tilde{\psi}_1}(x_{21}^-)(x_{23}^-)^r}\right)\right].
\end{aligned}
\tag{2.207}
$$

where we have substituted the expansion (2.199) and taken the $z\to 0$ limit. Finally, let us compute δS_4:

$$
\begin{aligned}
\delta S_4 = & -4\lim_{z\to 0}\int d^2x_0\sum_{r=1}^{2s-1}(2s-r-1)(r-1)\frac{1}{(x_{03}^-)^r}z^{r-s-2}(s+1) \\
& \times(\partial_{y^2})^{2s-r-2}(-\partial_{y^1})^{r-2}\delta C_{mat}^{(1),0}(x_{01})C_{mat}^{(1),2s-4}(x_{02}|y) \\
= & -4\lim_{z\to 0}\int d^2x_0\sum_{r=1}^{2s-1}(-1)^{r-2}\frac{1}{(x_{03}^-)^r}K_{01}^{1+\frac{\tilde{\psi}_1}{2}}K_{02}^{\frac{\tilde{\psi}_1}{2}+s-1}\frac{\psi_1^s}{(s-2)!}\left(1+(s-2)(1+\tilde{\psi}_1)\right) \\
& \times\sum_{u=0}^{s-2}\sum_{v=0}^{2u-r+2}\frac{(s-2)!(r-1)!(2s-r-1)!}{(s-u-2)!(2u-r+2-v)!(r-u-2)!v!}z^{2u-2v-s}(-x_{02}^-)^{r-u+v-2}(x_{02}^+)^{s-u+v-2}.
\end{aligned}
\tag{2.208}
$$

After substituting the boundary to bulk propagators and taking the $z\to 0$ limit, we obtain

$$
\begin{aligned}
\delta S_4 = & -4\sum_{r=1}^{2s-1}(s+1)\psi_1^s\left(1+(s-2)(1+\tilde{\psi}_1)\right) \\
& \times\left[\pi\frac{\Gamma(\frac{\tilde{\psi}_1}{2})}{\Gamma(1+\frac{\tilde{\psi}_1}{2})}\frac{1}{x_{12}^{\tilde{\psi}_1+2s-2}}\frac{(r-1)(2s-r-1)!}{(s-r)!}\frac{(x_{12}^-)^{r-2}(x_{12}^+)^{s-2}}{(x_{13}^-)^r}\right. \\
& +\pi\sum_{q=0}^{s-2}\sum_{u=0}^{s-2}\frac{\Gamma(s-2-q+\frac{\tilde{\psi}_1}{2})(r-1)!(2s-r-1)!q!}{\Gamma(s-1+\frac{\tilde{\psi}_1}{2})(s-u-2)!(u-r-q+s)!(r-u-2)!(q+u-s+2)!(s-r)!} \\
& \left.\times(-1)^{q-s}\partial_{x_2^-}^{s-r}\left(\frac{1}{x_{21}^{2+\tilde{\psi}_1}}\frac{1}{(x_{23}^-)^r}\right)\right].
\end{aligned}
\tag{2.209}
$$

The three point function is proportional to $\delta S=\delta S_1+U_1+U_3+\delta S_4$. One can simplify the above expressions and compute the full three point function directly, but since we are only interested in the overall coefficient whereas the position dependence is completely fixed by the conformal symmetry, we can take the limit in which one of the two scalar operators collides with the higher spin current, and extract the overall coefficient.

Let us define the variables $y_1^\pm = x_1^\pm - x_3^\pm$ and $y_2^\pm = x_2^\pm - x_3^\pm$, and consider the limit $y_1 \ll y_2$. The various pieces of contributions are given in this limit by

$$\begin{aligned}
\delta S_1 \to & 4(2-s)\psi_1^s\left(1+s(1+\tilde{\psi}_1)\right)2\pi\tilde{\psi}_1 s!\frac{1}{y_2^{2+\tilde{\psi}_1}}\frac{1}{(y_1^-)^s}, \\
U_1 \to & -4\psi_1^s\left(1+(s-1)(1+\tilde{\psi}_1)\right)(2+\tilde{\psi}_1)\frac{10\tilde{\psi}_1-8}{3}\pi(s-1)!\frac{1}{y_2^{\tilde{\psi}_1+2}}\frac{1}{(y_1^-)^s}, \\
U_3 \to & -4\psi_1^s\left(1+(s-1)(1+\tilde{\psi}_1)\right)(2+\tilde{\psi}_1)\frac{4\pi}{1+2\tilde{\psi}_1}s!\frac{1}{y_2^{2+\tilde{\psi}_1}}\frac{-s+1}{(y_1^-)^s}, \\
\delta S_4 \to & -4(s+1)\psi_1^s\left(1+(s-2)(1+\tilde{\psi}_1)\right)\pi\frac{\Gamma(\frac{\tilde{\psi}_1}{2})}{\Gamma(1+\frac{\tilde{\psi}_1}{2})}(s-1)(s-1)!\frac{1}{(y_1^-)^s}\frac{1}{y_2^{\tilde{\psi}_1+2}}.
\end{aligned} \tag{2.210}$$

Summing these four terms, and recovering the full position dependence using the conformal symmetry, we obtain the three point function of one higher spin current and two scalar operators:

$$\left\langle\left(\mathcal{O}+\overline{\mathcal{O}}\right)(x_1)\left(\mathcal{O}+\overline{\mathcal{O}}\right)(x_2)J^s(x_3)\right\rangle = 8\pi(s+\tilde{\psi}_1(s-1))(1+(-)^s)\Gamma(s)\frac{1}{|x_{12}|^{2+\tilde{\psi}_1}}\left(\frac{x_{12}^-}{x_{13}^-x_{23}^-}\right)^s. \tag{2.211}$$

Note that since we have turned on the sources for C_{even} so far, the dual scalar operator is $\mathcal{O}+\overline{\mathcal{O}}$. The three point function involving an insertion of $\mathcal{O}-\overline{\mathcal{O}}$, dual to the bulk field C_{odd}, can be computed analogously by turning on a source for C_{odd}. Note that C_{odd} is a purely imaginary field; in other words, if we write $C_{odd} = i\varphi$, then φ is a real field with the "right sign" kinetic term. A computation similar to the above gives

$$\left\langle \left(\mathcal{O}-\overline{\mathcal{O}}\right)(x_1)\left(\mathcal{O}+\overline{\mathcal{O}}\right)(x_2)J^s(x_3)\right\rangle = 8\pi(s+\tilde{\psi}_1(s-1))(1-(-)^s)\Gamma(s)\frac{1}{|x_{12}|^{2+\tilde{\psi}_1}}\left(\frac{x_{12}^-}{x_{13}^-x_{23}^-}\right)^s. \tag{2.212}$$

Adding (2.211) and (2.212), we obtain

$$\left\langle \overline{\mathcal{O}}(x_1)\mathcal{O}(x_2)J^s(x_3)\right\rangle = -4\pi(s+\tilde{\psi}_1(s-1))\Gamma(s)\frac{1}{|x_{12}|^{2+\tilde{\psi}_1}}\left(\frac{x_{12}^-}{x_{13}^-x_{23}^-}\right)^s. \tag{2.213}$$

2.C The deformed vacuum solution

In this section, we discuss the formulation of the three dimensional Vasiliev system as originally written in [22], which amounts to an extension of the equations (2.6) by introducing two additional auxiliary variables k and ρ, as described below, and the 1-parameter family of "deformed" vacuum solutions. The deformed vacuum solution of the system (2.6) can be obtain by a simple projection on the extended system. We will also present the boundary to bulk propagator for the B master field, which contains the bulk "matter" scalar field, in the deformed vacua, by solving the linearized equations.

To describe the deformed vacuum, it is useful to introduce two additional auxiliary variables k and ρ. They obey the following (anti-)commutation relations with one another and with the twistor variables (y,z):

$$k^2=\rho^2=1, \qquad \{k,\rho\}=\{k,y_\alpha\}=\{k,z_\alpha\}=0, \qquad [\rho,y_\alpha]=[\rho,z_\alpha]=0. \tag{2.214}$$

It will be also convenient to define the variable

$$w_\alpha=(z_\alpha+y_\alpha)\int_0^1 dt\, te^{tzy}. \tag{2.215}$$

It is straightforward to show that w_α satisfy the following star commutation relations:

$$\begin{aligned}
&[w_\alpha, w_\beta]_* = 0, \\
&[w_\alpha, y_\beta]_* + [y_\alpha, w_\beta]_* = 2\epsilon_{\alpha\beta} K, \\
&[w_\alpha, z_\beta]_* + [z_\alpha, w_\beta]_* = -2\epsilon_{\alpha\beta} K, \\
&\{w_\alpha, z_\beta\}_* * K - \{y_\alpha, w_\beta\}_* = 0.
\end{aligned} \tag{2.216}$$

Next, let us define

$$\begin{aligned}
&\tilde{z}_\alpha(\nu) = z_\alpha + \nu w_\alpha k, \\
&\tilde{y}_\alpha(\nu) = y_\alpha + \nu w_\alpha * K k.
\end{aligned} \tag{2.217}$$

Using the relations (2.216), it is easy to show that

$$\begin{aligned}
&[\tilde{y}_\alpha, \tilde{y}_\beta]_* = 2\epsilon_{\alpha\beta}(1 + \nu k), \\
&[\rho\tilde{z}_\alpha, \rho\tilde{z}_\beta]_* = -2\epsilon_{\alpha\beta}\left(1 + \nu K k\right), \\
&[\rho\tilde{z}_\alpha, \tilde{y}_\beta]_* = 0.
\end{aligned} \tag{2.218}$$

Under the star algebra, $\tilde{y}_\alpha$ generate the (deformed) three dimensional higher spin algebra $hs(\lambda)$ with $\lambda = \frac{1}{2}(1 + \nu k)$. Later we will make the projection onto the eigenspace of $k = 1$ or $k = -1$, in which case $\lambda = \frac{1}{2}(1 + \nu)$ or $\lambda = \frac{1}{2}(1 - \nu)$. The higher spin algebra $hs(\lambda)$ is an associative algebra, whose general element can be represented by an even analytic star-function in $\tilde{y}_\alpha$. In particular, it has an $sl(2)$-subalgebra whose generator can be written as $T_{\alpha\beta} = \tilde{y}_{(\alpha} * \tilde{y}_{\beta)}$.

The deformed vacuum solution is given by

$$\begin{aligned}
&B = \frac{1}{4}\nu, \qquad S_\alpha = \frac{1}{2}\rho(\tilde{z}_\alpha - z_\alpha), \\
&W = W_0 = w_0 + \psi_1 e_0 = \left(w_0^{\alpha\beta}(x) + \psi_1 e_0^{\alpha\beta}(x)\right) T_{\alpha\beta}.
\end{aligned} \tag{2.219}$$

They satisfy the (k,ρ)-extended Vasiliev equations:[10]

$$\begin{aligned}
&d_x W + W * W = 0,\\
&d_x S + d_z W + \{W, S\}_* = 0,\\
&d_z S + S * S = B * K k dz^2,\\
&d_z B + [S, B]_* = 0,\\
&d_x B + [W, B]_* = 0,
\end{aligned} \tag{2.220}$$

We can go back to the system (2.6) by simply multiplying a projector $\frac{1}{2}(1+k)$ on the left of every equation. Given any solution of the extended Vasiliev equations, by acting on it with the projector we obtain a solution of the equations (2.6). It follows that the deformed vacuum solution of (2.6) is

$$\begin{aligned}
&B = \frac{1}{4}\nu, \qquad S_\alpha = \frac{1}{2}\left(\tilde{z}_\alpha(-\nu) - z_\alpha\right),\\
&W = \left(w_0^{\alpha\beta}(x) + \psi_1 e_0^{\alpha\beta}(x)\right)\tilde{y}_\alpha(\nu) * \tilde{y}_\beta(-\nu).
\end{aligned} \tag{2.221}$$

Next, we will solve the linearize equation on the deformed vacua, and derive the boundary to bulk propagator for B (the scalar and corresponding auxiliary fields). For simplicity of the notation, we will work in the extended Vasiliev system. The boundary to bulk propagator for fields in the system (2.6) can be obtained simply by applying the projector $\frac{1}{2}(1+k)$. The linearized equations for B are

$$\begin{aligned}
&\left[\rho\tilde{z}_\alpha, B^{(1)}\right]_* = 0,\\
&D_0 B^{(1)} = 0.
\end{aligned} \tag{2.222}$$

where D_0 is defined by $D_0 \equiv d + [W_0, \cdot]$. The first equation of (2.222) immediately implies $B^{(1)}(x|y,z,\psi) = B_*^{(1)}(x|\tilde{y},\psi)$, where the subscript $*$ of a function means that it is a star-function.

[10]Note that the form of these equations differs from the system (2.6) only in the RHS of the third equation.

Decomposing $B_*^{(1)}(x|\tilde{y},\psi)$ as $B_*^{(1)}(x|\tilde{y},\psi) = C_{aux*}^{(1)}(x|\tilde{y},\psi_1) + \psi_2 C_{mat*}^{(1)}(x|\tilde{y},\psi_1)$, the second equation of (2.222) gives

$$\begin{aligned} &dC_{aux*}^{(1)} + [w_0, C_{aux*}^{(1)}]_* + \psi_1 [e_0, C_{aux*}^{(1)}]_* = 0, \\ &dC_{mat*}^{(1)} + [w_0, C_{mat*}^{(1)}]_* - \psi_1 \{e_0, C_{mat*}^{(1)}\}_* = 0. \end{aligned} \tag{2.223}$$

As in the case of equations in the undeformed vacuum analyzed in Section 2.3.1 and Appendix 2.A.1, the equation for $C_{aux*}^{(1)}$ is over-constraining, and eliminates all dynamical degrees of freedom of $C_{aux*}^{(1)}$. We will simply set $C_{aux*}^{(1)} = 0$, and only study the equation of the "matter" component $C_{mat*}^{(1)}$ in the following. Let us expand $C_{mat*}^{(1)}$ in the form

$$C_{mat*}^{(1)}(\tilde{y}) = \sum_{n=0}^{\infty} C_{mat*,\alpha_1\cdots\alpha_n}^{(1)} \tilde{y}^{(\alpha_1} * \cdots * \tilde{y}^{\alpha_n)}. \tag{2.224}$$

To compute the (anti-)commutators in (2.223), let us first consider the star product of $\tilde{y}^\alpha$ with $\tilde{y}^{(\alpha_1} * \cdots * \tilde{y}^{\alpha_n)}$:

$$\begin{aligned} &\tilde{y}^\alpha * \tilde{y}^{(\alpha_1} * \cdots * \tilde{y}^{\alpha_n)} \\ &= \tilde{y}^{(\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n)} + \frac{1}{n+1} \sum_{i=1}^{n} (n-i+1) \tilde{y}^{(\underline{\alpha_1}} * \cdots * [\tilde{y}^\alpha, \tilde{y}^{\underline{\alpha_i}}]_* * \cdots * \tilde{y}^{\underline{\alpha_n})} \\ &= \tilde{y}^{(\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n)} + \frac{1}{n+1} \sum_{i=1}^{n} (n-i+1)(1+(-)^{i-1}\nu k) 2\epsilon^{\alpha(\alpha_i} \tilde{y}^{\alpha_1} * \cdots * \cancel{\tilde{y}^{\alpha_i}} * \cdots * \tilde{y}^{\alpha_n)}. \end{aligned} \tag{2.225}$$

Contracting the above with $e_\alpha C_{\alpha_1\cdots\alpha_n}$ (here and in what follows, e and C are used to denote arbitrary totally symmetric tensors), we obtain

$$\begin{aligned} &e_\alpha \tilde{y}^\alpha * C_{\alpha_1\cdots\alpha_n} \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n} \\ &= e_{(\underline{\alpha}} C_{\underline{\alpha_1\cdots\alpha_n})} \tilde{y}^\alpha * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n} - a(n,\nu k) e^\alpha C_{\alpha\alpha_1\cdots\alpha_{n-1}} \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-1}}, \end{aligned} \tag{2.226}$$

where

$$a(n,\nu k) = 2\sum_{i=1}^{n} \frac{1}{(n+1)} (n-i+1)(1+(-)^{i-1}\nu k). \tag{2.227}$$

Applying a similar operation, staring $\tilde{y}^{(\alpha} * \tilde{y}^{\beta)}$ with $\tilde{y}^{(\alpha_1} * \cdots * \tilde{y}^{\alpha_n)}$ and contracting with $e_{\beta\alpha} C_{\alpha_1 \cdots \alpha_n}$, we get

$$
\begin{aligned}
&e_{\beta\alpha}\tilde{y}^{\beta} * \tilde{y}^{\alpha} * C_{\alpha_1\cdots\alpha_n}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n} = e_{(\underline{\beta\alpha}}C_{\underline{\alpha_1\cdots\alpha_n})}\tilde{y}^{\beta} * \tilde{y}^{\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n} \\
&\quad - \frac{n}{n+1}a(n+1,\nu k)e^{\beta}{}_{(\underline{\alpha}}C_{\beta\underline{\alpha_1\cdots\alpha_{n-1}})}\tilde{y}^{\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-1}} \\
&\quad - a(n,-\nu k)e_{(\underline{\beta}}{}^{\alpha}C_{\alpha\underline{\alpha_1\cdots\alpha_{n-1}})}\tilde{y}^{\beta} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-1}} \\
&\quad + a(n,-\nu k)a(n-1,\nu k)e^{\alpha\beta}C_{\alpha\beta\alpha_1\cdots\alpha_{n-2}}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-2}}.
\end{aligned}
\tag{2.228}
$$

Now, starring $\tilde{y}^{\alpha}$ with $\tilde{y}^{(\alpha_1} * \cdots * \tilde{y}^{\alpha_n)}$ from the right side,

$$
\begin{aligned}
&\tilde{y}^{(\alpha_1} * \cdots * \tilde{y}^{\alpha_n)} * \tilde{y}^{\alpha} \\
&= \tilde{y}^{(\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n)} + \frac{1}{n+1}\sum_{i=1}^{n}(-i)\tilde{y}^{(\underline{\alpha_1}} * \cdots * [\tilde{y}^{\alpha}, \tilde{y}^{\underline{\alpha_i}}]_* * \cdots * \tilde{y}^{\underline{\alpha_n})} \\
&= \tilde{y}^{(\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n)} + \frac{1}{n+1}\sum_{i=1}^{n}(-i)(1+(-)^{i-1}\nu k)2\epsilon^{\alpha(\alpha_i}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\cancel{\alpha_i}} * \cdots * \tilde{y}^{\alpha_n)}.
\end{aligned}
\tag{2.229}
$$

Contracting this formula with $e_{\alpha} C_{\alpha_1 \cdots \alpha_n}$, we have

$$
\begin{aligned}
&C_{\alpha_1\cdots\alpha_n}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n} * e_{\alpha}\tilde{y}^{\alpha} \\
&= e_{(\underline{\alpha}}C_{\underline{\alpha_1\cdots\alpha_n})}\tilde{y}^{\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n} - b(n,\nu k)e^{\alpha}C_{\alpha\alpha_1\cdots\alpha_{n-1}}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-1}},
\end{aligned}
\tag{2.230}
$$

where

$$
b(n,\nu k) = 2\sum_{i=1}^{n}\frac{1}{(n+1)}(-i)(1+(-)^{i-1}\nu k).
\tag{2.231}
$$

Performing a similar operation with $\tilde{y}^{(\alpha} * \tilde{y}^{\beta)}$, we obtain

$$
\begin{aligned}
&C_{\alpha_1\cdots\alpha_n}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n} * e_{\beta\alpha}\tilde{y}^{\beta} * \tilde{y}^{\alpha} = e_{(\underline{\beta\alpha}}C_{\underline{\alpha_1\cdots\alpha_n})}\tilde{y}^{\beta} * \tilde{y}^{\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n} \\
&\quad - \frac{n}{n+1}b(n+1,\nu k)e^{\beta}{}_{(\underline{\alpha}}C_{\beta\underline{\alpha_1\cdots\alpha_{n-1}})}\tilde{y}^{\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-1}} \\
&\quad - b(n,\nu k)e_{(\underline{\beta}}{}^{\alpha}C_{\alpha\underline{\alpha_1\cdots\alpha_{n-1}})}\tilde{y}^{\beta} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-1}} \\
&\quad + b(n,\nu k)b(n-1,\nu k)e^{\alpha\beta}C_{\alpha\beta\alpha_1\cdots\alpha_{n-2}}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-2}}.
\end{aligned}
\tag{2.232}
$$

Adding (2.228) and (2.232), we obtain the anticommutator:

$$\{e_{\beta\alpha}\tilde{y}^{\beta} * \tilde{y}^{\alpha}, C_{\alpha_1\cdots\alpha_n}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n}\}_* = 2e_{(\beta\underline{\alpha}}C_{\underline{\alpha_1\cdots\alpha_n})}\tilde{y}^{\beta} * \tilde{y}^{\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n}$$
$$+ f(n,\nu k)e^{\beta}{}_{(\underline{\alpha}}C_{\beta\underline{\alpha_1\cdots\alpha_{n-1}})}\tilde{y}^{\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-1}} + g(n,\nu k)e^{\alpha\beta}C_{\alpha\beta\alpha_1\cdots\alpha_{n-2}}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-2}}, \tag{2.233}$$

where

$$\begin{aligned} f(n,\nu k) &= -\frac{n}{n+1}a(n+1,\nu k) - a(n,-\nu k) - \frac{n}{n+1}b(n+1,\nu k) - b(n,\nu k), \\ g(n,\nu k) &= a(n,-\nu k)a(n-1,\nu k) + b(n,\nu k)b(n-1,\nu k). \end{aligned} \tag{2.234}$$

If n is even, $f(n,\nu k)$ and $g(n,\nu k)$ can be further simplified to

$$\begin{aligned} f(2j,\nu k) &= 0, \\ g(2j,\nu k) &= 4j\frac{(1+2j-\nu k)(-1+2j+\nu k)}{1+2j}. \end{aligned} \tag{2.235}$$

Subtracting (2.228) from (2.232), we obtain the commutator:

$$\left[w_{\beta\alpha}\tilde{y}^{\beta} * \tilde{y}^{\alpha}, C_{\alpha_1\cdots\alpha_n}\tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_n}\right]_* = -4nw^{\beta}{}_{(\underline{\alpha}}C_{\beta\underline{\alpha_1\cdots\alpha_{n-1}})}\tilde{y}^{\alpha} * \tilde{y}^{\alpha_1} * \cdots * \tilde{y}^{\alpha_{n-1}}. \tag{2.236}$$

The linearized equation (2.223) for the matter field, therefore, can be written as

$$\begin{aligned} &\partial_\mu C^{(1),n}_{mat\ \alpha_1\cdots\alpha_n} - 4n(w_{0\mu})_{(\underline{\alpha_1}}{}^{\beta}C^{(1),n}_{mat\ \beta\underline{\alpha_2\cdots\alpha_n})} - 2\psi_1(e_{0\mu})_{(\underline{\alpha_1\alpha_2}}C^{(1),n-2}_{mat\ \ \underline{\alpha_3\cdots\alpha_n})} \\ &\qquad - g(n+2,\nu k)\psi_1(e_{0\mu})^{\alpha\beta}C^{(1),n+2}_{mat\ \ \alpha\beta\alpha_1\cdots\alpha_n} = 0. \end{aligned} \tag{2.237}$$

After contracting with $(e_0^{\mu})_{\alpha\beta}$, this equation is written as

$$\nabla_{\alpha\beta}C^{(1),n}_{mat\ \alpha_1\cdots\alpha_n} + \frac{1}{16}\psi_1\epsilon_{(\alpha(\underline{\alpha_1}}\epsilon_{\beta)\underline{\alpha_2}}C^{(1),n-2}_{mat\ \ \underline{\alpha_3\cdots\alpha_n})} + \frac{1}{32}g(n+2,\nu k)\psi_1C^{(1),n+2}_{mat\ \ \alpha\beta\alpha_1\cdots\alpha_n} = 0. \tag{2.238}$$

We follow the same procedure used in analyzing the undeformed vacuum, decomposing the above equation according to the action of permutation group on the indices. Contracting (2.238) with $\epsilon^{\alpha\alpha_1}$ gives

$$\nabla^{\alpha}{}_{\beta}C^{(1),n}_{mat\ \alpha\alpha_2\cdots\alpha_n} - \frac{n+1}{16n}\psi_1\epsilon_{\beta(\underline{\alpha_2}}C^{(1),n-2}_{mat\ \ \underline{\alpha_3\cdots\alpha_n})} = 0. \tag{2.239}$$

Further contracting (2.239) with $\epsilon^{\beta\alpha_2}$ gives

$$\nabla^{\alpha\beta} C^{(1),n}_{mat\ \alpha\beta\alpha_3\cdots\alpha_n} + \frac{n+1}{16(n-1)}\psi_1 C^{(1),n-2}_{mat\ \ \alpha_3\cdots\alpha_n} = 0. \tag{2.240}$$

As in the analysis of undeformed vacuum, now contracting the indices of the equations (2.238), (2.239), and (2.240) with the y^α's, we obtain

$$\begin{aligned} &\nabla^+ C^{(1),n}_{mat}(y) - \frac{1}{32} g(n+2,\nu k)\psi_1 C^{(1),n+2}_{mat}(y) = 0, \\ &\nabla^0 C^{(1),n}_{mat}(y) = 0, \\ &\nabla^- C^{(1),n}_{mat}(y) - \frac{1}{16}(n+1)n\psi_1 C^{(1),n-2}_{mat}(y) = 0, \end{aligned} \tag{2.241}$$

where

$$C^{(1),n}_{mat}(y) \equiv C^{(1),n}_{mat\ \alpha_1\cdots\alpha_n} y^{\alpha_1}\cdots y^{\alpha_n}. \tag{2.242}$$

Iterating the first equation of (2.241), we obtain

$$C^{(1),2s}_{mat}(y) = \left(\prod_{j=1}^{s} \frac{1}{g(2j,\nu k)}\right)(32\psi_1\nabla^+)^s C^{(1),0}_{mat}. \tag{2.243}$$

Since $C^{(1)}_{mat}(y)$ is restricted to be even in y^α, it is entirely determined by the bottom component $C^{(1),0}_{mat}$ via the above relation. After some simple manipulations of (2.241) using (2.106), we derive the second order form linearized equation

$$\Box_{AdS} C^{(1),n}_{mat} = -\frac{1}{8}\left(4n+8+\frac{n+1}{n}g(n,\nu k)\right) C^{(1),n}_{mat}. \tag{2.244}$$

For $n=0$, the equation is just the usual Klein-Gordon equation on AdS$_3$, and can be rewritten in a more familiar form:

$$\left(\nabla^\mu\partial_\mu - m^2\right) C^{(1),0}_{mat} = 0, \quad m^2 = -\frac{1}{4}(3-\nu k)(1+\nu k). \tag{2.245}$$

Depending on the choice of AdS boundary condition, this scalar field is dual to an operator of dimension

$$\Delta_\pm = 1 \pm \frac{1-\nu k}{2} = \frac{1+\nu k}{2} \ \text{ or } \ \frac{3-\nu k}{2}. \tag{2.246}$$

It is convenient to package the choice of boundary condition into a variable $\tilde{\psi}_1$, obeying $\tilde{\psi}_1^2 = 1$, so that the scaling dimension of the dual operator can be written as

$$\Delta = 1 + \tilde{\psi}_1 \left(\frac{1 - \nu k}{2} \right). \tag{2.247}$$

The boundary to bulk propagator for the scalar field is a solution of (2.245), which up to normalization is given by

$$C_{mat}^{(1),0} = K^{\Delta}, \quad \text{where} \quad K = \frac{z}{\vec{x}^2 + z^2}. \tag{2.248}$$

Here $(\vec{x}, z)$ are Poincaré coordinates of the AdS$_3$ (not to be confused with the twistor variable z_α). Using (2.109) and (2.243), we obtain

$$\begin{aligned} C_{mat}^{(1)}(y) &= \sum_{s=0}^{\infty} C_{mat}^{(1),2s}(y) \\ &= \sum_{s=0}^{\infty} \left(\prod_{j=1}^{s} \frac{\Delta + j - 1}{g(2j, \nu k)} \right) (4\psi_1)^s (y \Sigma y)^s K^{\Delta} \\ &= \sum_{s=0}^{\infty} \left(\prod_{j=1}^{s} \frac{(\Delta + j - 1)(1 + 2j)}{j(1 + 2j - \nu k)(-1 + 2j + \nu k)} \right) \psi_1^s (y \Sigma y)^s K^{\Delta} \\ &= {}_1F_1 \left(\frac{3}{2}, 1 - \tilde{\psi}_1 \left(\frac{1 - \nu k}{2} \right), \frac{1}{2} \psi_1 y \Sigma y \right) K^{1 + \tilde{\psi}_1 \left(\frac{1 - \nu k}{2} \right)}. \end{aligned} \tag{2.249}$$

In the actual master field, the above expression should be understood as a star-function, with y replaced by $\tilde{y}$. More concretely, we can transform the ordinary function $C_{mat}^{(1)}(y)$ to the star-function $C_{mat*}^{(1)}(\tilde{y})$ via the formula

$$C_{mat*}^{(1)}(\tilde{y}) = \frac{1}{(2\pi)^2} \int d^2y d^2u \, C_{mat}^{(1)}(y) e^{iuy} \exp_*(-iu\tilde{y}). \tag{2.250}$$

Chapter 3

Correlators in W_N Minimal Model Revisited

3.1 Introduction

The AdS/CFT correspondence [1, 2, 3] is one of the most important insights that came out of the study of string theory. While it is often said that both strings and the holographic dimension emerge from the large N and strong 't Hooft coupling limit of a gauge theory, there are really two separate dualities in play here. Firstly, a large N CFT, regardless of whether the 't Hooft coupling is weak or strong, is holographically dual to some theory of gravity together with higher spin fields in AdS, whose coupling is controlled by $1/N$ [19]. It often happens that, then, as a 't Hooft coupling parameter varies from weak to strong, the bulk theory interpolates between a higher spin gauge theory and a string theory (where the AdS radius becomes finite or large in string units). The duality as two separate stories: holography from large N, and the emergence of strings out of bound states of higher spin fields, has become particularly evident in [21].

The holographic dualities between higher spin gauge theories in AdS and vector model CFTs [19, 20, 4, 21] are a nice class of examples in that they avoid the complication of the second story mentioned above.[1] Both sides of the duality can be studied order by order in the $1/N$ expansion. The $\mathrm{AdS}_3/\mathrm{CFT}_2$ version of this duality, proposed by Gaberdiel and Gopakumar [4], relates a higher spin gauge theory coupled to scalar matter fields in AdS_3 [22] and the W_N minimal model in two dimensions [31].[2] While it was proposed in [4] that the bulk theory is Vasiliev's system in AdS_3, it was pointed out in [10] and in [12] that Vasiliev's system should be dual only perturbatively in $1/N$ to a subsector of the W_N minimal model, while the full non-perturbative duality requires adding new perturbative states in the bulk.[3]

One of the key observations of [4] is that the $W_{N,k}$ minimal model has a 't Hooft-like limit, where N is taken to be large while the "'t Hooft coupling" $\lambda = \frac{N}{k+N}$ is held finite. The basic evidence is that the spectrum of operators organize into that of "basic primaries", which are dual to elementary particles in the bulk, and the composite operators which are dual to bound states of elementary particles. It was not obvious, however, that the correlation functions obey large N factorization, as for single trace operators in large N gauge theories. This will be demonstrated in the current paper. In particular, we will understand which operators are the fundamental particles, and which ones are their bound states, by extracting

[1] See [33, 34, 35, 36, 23, 37] for recent nontrivial checks and progress toward deriving the duality with vector models.

[2] For works leading up to this duality, and explorations on its consequences, see [5, 6, 7, 38, 39, 30, 24, 10, 12, 40, 41, 42].

[3] See [42] however for intriguing candidates for some new bulk states in higher spin gauge theories in AdS_3.

such information from the $1/N$ expansion of exact correlation functions in the W_N minimal model.

Our main findings are summarized as follows.

1. We derive all sphere three point functions of primaries in the W_N minimal model of the following form: one of the primaries is labelled by a pair of $SU(N)$ representations (Λ_+, Λ_-), both of which are symmetric products of the fundamental (or anti-fundamental) representation $\mathbf{f}$ (or $\bar{\mathbf{f}}$), and the other two primaries are completely general.[4] We see the explicit large N factorization in these three point functions. For example, denote by ϕ the primary $(\mathbf{f}, 0)$ (on both left and right moving sector). The large N factorization leads to the identification

$$\begin{aligned} (A,0) &\sim \frac{1}{\sqrt{2}}\phi^2, \\ (S,0) &\sim \frac{1}{\sqrt{2}\Delta_{(\mathbf{f},0)}}(\phi\partial\bar{\partial}\phi - \partial\phi\bar{\partial}\phi), \end{aligned} \tag{3.1}$$

where A and S are the anti-symmetric and symmetric tensor product representation of $\mathbf{f}$, and $\Delta_{(\mathbf{f},0)} = 1 + \lambda$ is the scaling dimension of ϕ at large N. This large N factorization is a simple check of the duality, in verifying that $(A, 0)$ and $(S, 0)$ are indeed bound states of two elementary scalar particles in the bulk, and behave as two free particles in the infinite N limit.

A less obvious example concerns the "light" primary $(\mathbf{f}, \mathbf{f})$, which we denote by ω. Its scaling dimension $\Delta_{(\mathbf{f},\mathbf{f})}$ vanishes in the infinite N limit, and is given by $\Delta_{(\mathbf{f},\mathbf{f})} = \lambda^2/N$ at order $1/N$. Two candidates for the lowest bound state of two ω's are (A, A) and (S, S),

[4] The technique used in this paper allows us to go beyond this set using four point functions, but we will not present those results here.

both of which have scaling dimension $2\Delta_{(\mathbf{f},\mathbf{f})}$ at order $1/N$. We will find that

$$\frac{(A,A)+(S,S)}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}\omega^2 \tag{3.2}$$

is the bound state of two ω's, while $\frac{1}{\sqrt{2}}((A,A)-(S,S))$ is a new elementary light particle in the bulk. This shows that the elementary light particles in the bulk also interact weakly in the large N limit.

A word of caution is that even in the infinite N limit, the space of states is *not* the freely generated Fock space of single particle primary states and their descendants. As observed in [12], for instance, the level $(1,1)$ descendant of ω, namely $\frac{1}{\Delta_{(\mathbf{f},\mathbf{f})}}\partial\bar{\partial}\omega$, should be identified with the the two-particle state (or "double trace operator") $\phi\,\widetilde{\phi}$, where $\widetilde{\phi}$ is the other basic primary $(0,\mathbf{f})$. We will see that this identification is consistent with the large N factorization of composite operators made out of ω, ϕ, and $\widetilde{\phi}$. This suggests that the Hilbert space at infinite N is a quotient of the freely generated Fock space, with identifications such as $\frac{1}{\Delta_{(\mathbf{f},\mathbf{f})}}\partial\bar{\partial}\omega \sim \phi\widetilde{\phi}$. This peculiar feature is closed tied to the presence of light states. The large N factorization in the W_N minimal model holds only up to such identifications.

2. We compute the sphere four-point function of $(\mathbf{f},0)$, $(\bar{\mathbf{f}},0)$, with a general primary (Λ_+,Λ_-) and its charge conjugate, which generalizes the four-point functions considered in [12]. This result is not new and is in fact contained in [43]. In [43], the sphere four-point function was obtained by solving the differential equation due to a null state, which we will review. The method gives the answer for general N, but is not easy to generalize to correlators on a Riemann surface of nonzero genus. We will then consider an alternative method, using contour integrals of screening charges. This second method requires knowing which contours correspond to which conformal blocks; they will be analyzed in detail through the investigation of monodromies. While this approach appears rather cumbersome due to

the complexity of the contour integral, it allows for a straightforward generalization to the computation of torus two-point functions.[5]

3. We derive a contour integral expression for the torus two-point function of the basic primaries $(\mathbf{f}, 0)$ and $(\bar{\mathbf{f}}, 0)$. Since the result is exact, it can be analytically continued to Lorentzian signature, yielding the Lorentzian thermal two-point function on the circle. The latter is a useful probe of the dual bulk geometry. In a theory of ordinary gravity in AdS_3, at temperatures above the Hawking-Page transition, the dominant phase is the BTZ black hole. The thermal two-point function on the boundary should see the thermalization of the black hole reflected in an exponential decay behavior of the correlator, for a very long time before Poincaré recurrence kicks in.[6] While the BTZ black hole clearly exists in any higher spin gravity theory in AdS_3, it is unclear whether the BTZ black hole will be the dominant phase at any temperature at all, as there can be competing higher spin black hole solutions (see [40, 41, 42]). Nonetheless, the question of whether thermalization occurs at the level of two-point functions can be answered definitively using the exact torus two-point function. So far, it appears to be difficult to extract the large N behavior from our exact contour integral expression, which we leave to future work. In the $N = 2$ case, i.e. Virasoro minimal model, where the contour integral involved is a relatively simple one, we computed numerically certain thermal two-point functions at integer values of times, as a demonstration in principle.

[5] Our method is a direct generalization of [44], where the torus two-point function in the Virasoro minimal model was derived.

[6] In the W_N minimal model, all scaling dimensions are integer multiples of $\frac{1}{N(N+k)(N+k+1)} \sim \frac{\lambda^2}{N^3}$, and hence Poincaré recurrence must already occur at no later than time scale N^3. In fact, we will see that the Poincare recurrence in the two-point function under consideration occurs at an even shorter time $N(k+N)$. But if the BTZ black hole dominates the bulk in some temperature of order 1, we should expect to see thermalization at time scale of order 1 (and $\ll N^2$).

In Section 3.2, we will summarize the definitions and convention for W_N minimal model which will be used throughout this paper. Section 3.3 describes the strategy of the computation, namely using the Coulomb gas formalism. In Section 3.4, 3.5, 3.6 we present a class of sphere three, four-point, and torus two-point functions, make various checks of the result, and discuss the implications. We conclude in Section 3.7.

3.2 Definitions and conventions for the W_N minimal model

The W_N minimal model can be realized as the coset model

$$\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}}. \tag{3.3}$$

A priori, through the coset construction, the W_N primaries are labeled by a triple of representations of $SU(N)$ current algebra $(\rho, \mu; \nu)$ (at level $k, 1$, and $k+1$ respectively.) By a slight abuse of notation, we will also denote by ρ, μ, ν the corresponding highest weight vectors. The three representations are subject to the constraint that $\rho + \mu - \nu$ lies in the root lattice of $SU(N)$. Each representation is subject to the condition that the sum of $N-1$ Dynkin labels is less than or equal to the affine level. This condition determines μ uniquely, given ρ and ν. We will therefore label the primaries by the pair of the representations $(\rho; \nu) \equiv (\Lambda_+, \Lambda_-)$ from now on.

Let α_i, $i = 1, \cdots, N-1$, be the simple roots of $SU(N)$. They have inner product $\alpha_i \cdot \alpha_j = A_{ij}$, where A_{ij} is the Cartan matrix. In particular, $\alpha_i^2 = 2$. Let ω^i, $i = 1, \cdots, N-1$, be the fundamental weights. They obey $\omega^i \cdot \alpha_j = \delta^i_j$. We write $F^{ij} = \omega^i \cdot \omega^j = (A^{-1})^{ij}$. The highest weight λ of some representation Λ takes the form

$$\lambda = \sum_{i=1}^{N-1} \lambda_i \omega^i, \tag{3.4}$$

where $(\lambda_1, \cdots, \lambda_{N-1}) \in \mathbb{Z}_{\geq 0}^{N-1}$ are the Dynkin labels.

The Weyl vector is

$$\rho = \sum_{i=1}^{N-1} \omega^i, \tag{3.5}$$

i.e. it has Dynkin label $(1, 1, \cdots, 1)$.

Given a root α, the simple Weyl reflection with respect to α acts on a weight λ by

$$s_\alpha(\lambda) = \lambda - (\alpha \cdot \lambda)\alpha. \tag{3.6}$$

A general Weyl group element w can be written as $w = s_{\alpha_1} \cdots s_{\alpha_m}$. We will use the notation $w(\lambda)$ for the Weyl reflection of λ by w. The *shifted* Weyl reflection $w \cdot \lambda$ is defined by

$$w \cdot \lambda = w(\lambda + \rho) - \rho. \tag{3.7}$$

Now let us discuss the W_N character of a primary (Λ_+, Λ_-). Throughout this paper, we use the notation $p = k + N$ and $p' = k + N + 1$. The central charge is

$$c = N - 1 - \frac{N(N^2-1)}{pp'}. \tag{3.8}$$

Note that $\rho^2 = \frac{1}{12}N(N^2-1)$. The conformal dimension of the primary is

$$h_{(\Lambda_+,\Lambda_-)} = \frac{1}{2pp'}\left(|p'\Lambda_+ - p\Lambda_- + \rho|^2 - \rho^2\right). \tag{3.9}$$

The character of (Λ_+, Λ_-) can be written as a sum over *affine* Weyl group elements,

$$\chi^N_{(\Lambda_+,\Lambda_-)}(\tau) = \frac{1}{\eta(\tau)^{N-1}} \sum_{\hat{w} \in \widehat{W}} \epsilon(\hat{w}) q^{\frac{1}{2pp'}|p'\hat{w}(\Lambda_+ + \rho) - p(\Lambda_- + \rho)|^2}, \tag{3.10}$$

where $\widehat{W}$ is given by the semi-direct product of W with translations by p times the root lattice, namely an element $\hat{w} \in \widehat{W}$ acts on a weight vector λ by

$$\hat{w}(\lambda) = w(\lambda) + pn^i\alpha_i, \qquad w \in W,\ n_i \in \mathbb{Z}. \tag{3.11}$$

$\epsilon(\hat{w}) = \epsilon(w)$ is the signature of $\hat{w}$.

Let us illustrate this formula with the $N = 2$ example, i.e. Virasoro minimal model. Write $\Lambda_+ = (r-1)\omega^1$, $1 \le r \le p-1 = k+1$, and $\Lambda_- = (s-1)\omega^1$, $1 \le s \le p = k+2$. The Weyl group $\mathbb{Z}_2$ contains the reflection $w(\lambda) = -\lambda$. We have $\hat{w}(\Lambda_+ + \rho) = -r\omega^1 + pn\alpha_1 = (-r+2pn)\omega^1$. So

$$h_{r,s} = \frac{(p'r-ps)^2-1}{4pp'}, \tag{3.12}$$

and

$$\begin{aligned}\chi_{r,s}(\tau) &= \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}}\left[q^{\frac{1}{4pp'}(p'(r+2pn)-ps)^2} - q^{\frac{1}{4pp'}(p'(-r+2pn)-ps)^2}\right] \\ &= \frac{q^{\frac{1}{4pp'}(p'r-ps)^2}}{\eta(\tau)}\sum_{n\in\mathbb{Z}}\left[q^{n(pp'n+p'r-ps)} - q^{(pn-r)(p'n-s)}\right]\end{aligned} \tag{3.13}$$

The term corresponding to $(w, n=0)$ comes from the null state at level rs.

3.3 Coulomb gas formalism

The idea of Coulomb gas formalism is to represent operators in the W_N minimal model by vertex operators constructed out of $N-1$ free bosons. This allows for the construction of all W_N currents as well as the primaries of the correct scaling dimensions. However, the free boson correlators by themselves do not obey the correct fusion rules of the W_N minimal model. To obtain the correct correlation functions, suitable screening operators must be inserted, and integrated along contours in a conformally invariant manner. More precisely, one obtains in this way the W_N conformal blocks. One then needs to sums up the conformal blocks with coefficients determined by monodromies, etc. This strategy is explained below.

3.3.1 Rewriting free boson characters

Let us begin with the following character of $N-1$ free bosons, twisted by an $SU(N)$ weight vector λ,

$$\begin{aligned} \widetilde{K}^N_\lambda(\tau) &= \frac{1}{\eta(\tau)^{N-1}} \sum_{\alpha\in\Lambda_{root}} q^{\frac{1}{2pp'}|\lambda+pp'\alpha|^2} \\ &= \frac{1}{\eta(\tau)^{N-1}} \sum_{(n^1,\cdots,n^{N-1})\in\mathbb{Z}^{N-1}} q^{\frac{1}{2pp'}|\lambda+pp'n^j\alpha_j|^2}. \end{aligned} \tag{3.14}$$

Define the lattice

$$\Gamma_x = \sqrt{x}\Lambda_{root}, \tag{3.15}$$

and its dual lattice

$$\Gamma^*_x = \frac{1}{\sqrt{x}}\Lambda_{weight}. \tag{3.16}$$

We may then write

$$K^N_u(\tau) = \frac{1}{\eta(\tau)^{N-1}} \sum_{n\in\Gamma_{pp'}} q^{\frac{1}{2}(u+n)^2} \tag{3.17}$$

for $u\in\Gamma^*_{pp'}$. In fact, u may be defined in the quotient of lattices,

$$u\in\Gamma^*_{pp'}/\Gamma_{pp'}. \tag{3.18}$$

Note that the number of elements in $\Gamma^*_{pp'}/\Gamma_{pp'}$ is

$$\det(pp'A_{ij}) = N(pp')^{N-1}. \tag{3.19}$$

It is useful to consider the decomposition

$$u=\lambda+\lambda', \quad \lambda\in\Gamma^*_{\frac{p}{p'}}/\Gamma_{pp'},\ \lambda'\in\Gamma^*_{\frac{p'}{p}}/\Gamma_{pp'}. \tag{3.20}$$

This decomposition is well defined up to the identification

$$(\lambda,\lambda')\sim(\lambda+t,\lambda'-t),\quad t\in\Gamma^*_{\frac{1}{pp'}}/\Gamma_{pp'}=(\Gamma^*_{\frac{p}{p'}}\cap\Gamma^*_{\frac{p'}{p}})/\Gamma_{pp'}. \tag{3.21}$$

Consider the action of a simple Weyl reflection on $v\in\Gamma^*_x$,

$$w_\alpha(v)=v-(\alpha\cdot v)\alpha, \tag{3.22}$$

where α is a root. Since $(\alpha\cdot v)\alpha\in x^{-\frac{1}{2}}\Lambda_{root}=\Gamma_{\frac{1}{x}}$, the Weyl action is trivial on $\Gamma^*_x/\Gamma_{\frac{1}{x}}$. In particular, the Weyl action on u is trivial on $\Gamma^*_{\frac{1}{pp'}}/\Gamma_{pp'}$, and is well defined on λ and λ' separately. Therefore, one can define the *double* Weyl action by $W\times W$ on λ and λ' independently. This will be important in describing W_N primaries.

Now consider $N-1$ free bosons compactified on the Narain lattice $\Gamma^{N-1,N-1}$, which is even, self-dual, of signature $(N-1,N-1)$, defined as[7]

$$\Gamma^{N-1,N-1}=\{(v,\bar v)|v,\bar v\in\Gamma^*_{pp'},v-\bar v\in\Gamma_{pp'}\}. \tag{3.24}$$

The free boson partition function can be decomposed in terms of the characters as

$$Z^{bos}_{\Gamma^{N-1,N-1}}(\tau,\bar\tau)=\sum_{u\in\Gamma^*_{pp'}/\Gamma_{pp'}}|K^N_u(\tau)|^2. \tag{3.25}$$

[7] To see that $\Gamma^{N-1,N-1}$ is even, note that

$$(v,\bar v)\cdot(v,\bar v)=v^2-\bar v^2=v^2-(v+n)^2=-2v\cdot n-n^2, \tag{3.23}$$

where $n\in\Gamma_{pp'}$, and the RHS is an even integer. To see that it is self-dual, take a basis (v^i,v^i) and $(v_i,0)$, $i=1,\cdots,N-1$, where $v_i\in\Gamma_{pp'}$ and $v^i\in\Gamma^*_{pp'}$ are dual basis for the respective lattices. This basis is unimodular.

3.3.2 W_N characters and partition function

Consider a W_N primary (Λ_+, Λ_-). Using the decomposition $u = \lambda + \lambda'$ described in the previous subsection, we may rewrite the W_N character

$$\chi^N_{(\Lambda_+,\Lambda_-)}(\tau) = \frac{1}{\eta(\tau)^{N-1}} \sum_{\hat{w}\in\widehat{W}} \epsilon(\hat{w}) q^{\frac{1}{2pp'}|p'\hat{w}(\Lambda_+ + \rho) - p(\Lambda_- + \rho)|^2} \tag{3.26}$$

in the form $\chi^N_{\lambda+\lambda'}(\tau)$, where

$$\lambda = \sqrt{\frac{p'}{p}}(\Lambda_+ + \rho) \in \Gamma^*_{\frac{p}{p'}}, \qquad \lambda' = -\sqrt{\frac{p}{p'}}(\Lambda_- + \rho) \in \Gamma^*_{\frac{p'}{p}}. \tag{3.27}$$

In other words, we write

$$\begin{aligned} \chi^N_{\lambda+\lambda'}(\tau) &= \frac{1}{\eta(\tau)^{N-1}} \sum_{w\in W, n\in\Gamma_{pp'}} \epsilon(w) q^{\frac{1}{2}|w(\lambda)+\lambda'+n|^2} \\ &= \sum_{w\in W} \epsilon(w) K^N_{w(\lambda)+\lambda'}(\tau). \end{aligned} \tag{3.28}$$

The rationale for the alternating sum in the above formula is the following. The dimension of the free boson vertex operator $e^{i(u-Q)\cdot X}$ corresponding to the character K^N_u, with linear dilaton (as will be described in the next subsection), is

$$h_u = \frac{1}{2}u^2 - \frac{1}{2}Q^2. \tag{3.29}$$

Let w be a simple Weyl reflection, by a root α_w. A simple computation shows that

$$h_{w(\lambda)+\lambda'} = h_{\lambda+\lambda'} + (\alpha_w \cdot \lambda)(-\alpha_w \cdot \lambda'). \tag{3.30}$$

If we restrict λ and $-\lambda'$ to sit in the identity Weyl chamber of $\Gamma^*_{\frac{p}{p'}}$ and $\Gamma^*_{\frac{p'}{p}}$, then $(\alpha_w \cdot \lambda)(-\alpha_w \cdot \lambda')$ is always a nonnegative integer. It is possible to subtract off the character $K^N_{w(\lambda)+\lambda'}$ to make the theory "smaller". The alternating sum in (3.28) does this in a Weyl invariant manner[8], and gives the character $\chi^N_{\lambda+\lambda'}(\tau)$ of the W_N minimal model.

[8] For w not a simple Weyl reflection, one can show that $h_{w(\lambda)+\lambda'} - h_{\lambda+\lambda'}$ is still a nonnegative integer, when λ and $-\lambda'$ sit in the identity Weyl chamber of $\Gamma^*_{\frac{p}{p'}}$ and $\Gamma^*_{\frac{p'}{p}}$.

Note that $\chi^N_{\lambda+\lambda'}(\tau)$ vanishes identically whenever (λ, λ') is fixed by the action of a subgroup of the double Weyl group $W \times W$. The set of inequivalent characters are thus parameterized by

$$E = (\Gamma^*_{pp'}/\Gamma_{pp'} - \{\text{fixed points}\})/W \times W. \tag{3.31}$$

This is also the set of inequivalent W_N primaries. The partition function of the W_N minimal model is given by the diagonal modular invariant

$$\begin{aligned} Z^N_{p,p'}(\tau,\bar\tau) &= \sum_{(\Lambda_+,\Lambda_-)} |\chi^N_{(\Lambda_+,\Lambda_-)}(\tau)|^2 \\ &= \frac{1}{N(N!)^2} \sum_{\lambda\in\Gamma^*_{\frac{p}{p'}}/\Gamma_{pp'},\ \lambda'\in\Gamma^*_{\frac{p'}{p}}/\Gamma_{pp'}} |\chi^N_{\lambda+\lambda'}(\tau)|^2 \\ &= \frac{1}{(N!)^2} \sum_{u\in\Gamma^*_{pp'}/\Gamma_{pp'}} |\chi^N_u(\tau)|^2, \end{aligned} \tag{3.32}$$

where the first sum is only over inequivalent (Λ_+, Λ_-) under shifted Weyl reflections. The decomposition $u = \lambda + \lambda'$ is understood in going between the last two lines (λ, λ' are defined up to a shift by $t \in \Gamma^*_{\frac{1}{pp'}}/\Gamma_{pp'}$).

Let us illustrate again with the $N = 2$ example. In this case, $\Gamma_{pp'} = \sqrt{2pp'}\,\mathbb{Z}$, $\Gamma^*_{pp'} = \frac{1}{\sqrt{2pp'}}\mathbb{Z}$. We have

$$\lambda \in \sqrt{\frac{p'}{2p}}\,\mathbb{Z}, \quad \lambda' \in \sqrt{\frac{p}{2p'}}\,\mathbb{Z}, \quad t \in \sqrt{\frac{pp'}{2}}\,\mathbb{Z}, \tag{3.33}$$

and

$$\frac{\Gamma^*_{pp'}}{\Gamma_{pp'}} \simeq \frac{\mathbb{Z}_{2p} \times \mathbb{Z}_{2p'}}{\mathbb{Z}_2} \tag{3.34}$$

$W \simeq \mathbb{Z}_2$ acts on Γ_x by reflection. The set of inequivalent characters is

$$E \simeq \frac{\mathbb{Z}^\times_p \times \mathbb{Z}^\times_{p'}}{\mathbb{Z}_2}, \tag{3.35}$$

where the $\mathbb{Z}_2$ identification on $\mathbb{Z}_p^\times \times \mathbb{Z}_{p'}^\times$ is

$$(r,s) \to (r+p, s+p') \sim (p-r, p'-s). \tag{3.36}$$

Returning to the general W_N characters, the modular transformation on $\chi_u^N(\tau)$ takes the form

$$\chi_u^N(-1/\tau) = \sum_{\tilde{u}\in\Gamma^*_{pp'}/\Gamma_{pp'}} \tilde{S}_{u,\tilde{u}}\chi_{\tilde{u}}^N(\tau), \qquad \tilde{S}_{u,\tilde{u}} = \frac{1}{\sqrt{N(pp')^{N-1}}} e^{-2\pi i u\cdot\tilde{u}}. \tag{3.37}$$

The RHS is not yet written as a sum over independent characters. After doing so, we have

$$\chi_u^N(-1/\tau) = \sum_{\tilde{u}\in\left(\Gamma^*_{pp'}/\Gamma_{pp'}-\text{fixed}\right)/W\times W} S_{u,\tilde{u}}\chi_{\tilde{u}}^N(\tau), \tag{3.38}$$

where

$$S_{u,\tilde{u}} = \sum_{(w,w')\in W\times W} \epsilon(w)\epsilon(w') S_{u,w(\tilde{\lambda})+w'(\tilde{\lambda}')}. \tag{3.39}$$

3.3.3 Coulomb gas representation of vertex operators and screening charge

We have seen that the partition function of the W_N minimal model may be obtained from that of the free bosons on the lattice $\Gamma^{N-1,N-1}$ by twisting by $\epsilon(w)$ in a sum over action by Weyl group elements $w \in W$. The free boson vertex operators corresponding to lattice vectors of $\Gamma^{N-1,N-1}$ take the form

$$e^{iv\cdot X + i\beta\cdot X_L}, \tag{3.40}$$

where $v \in \Gamma^*_{pp'}$, and $\beta \in \Gamma_{pp'}$. The lowest weight states appearing in the characters $|K_u^N|^2$ are of the form $e^{iv\cdot X}$.

Given a W_N primary labeled by (Λ_+, Λ_-), we associate it with the free boson vertex operator $e^{iv\cdot X}$, with the identification

$$v = \sqrt{\frac{p'}{p}}\Lambda_+ - \sqrt{\frac{p}{p'}}\Lambda_-. \tag{3.41}$$

In order to match the conformal dimensions, we need to turn on a linear dilaton background charge $Q = 2v_0\rho$, where $v_0 = \frac{1}{2}\left(\sqrt{\frac{p}{p'}} - \sqrt{\frac{p'}{p}}\right) = -\frac{1}{2\sqrt{pp'}}$. The conformal weight of $e^{iv\cdot X}$ in the linear dilaton CFT is then

$$h_{v-Q} = \frac{1}{2}(v-Q)^2 - \frac{1}{2}Q^2 = \frac{1}{2}v^2 - Q\cdot v. \tag{3.42}$$

Using

$$\begin{aligned} u = v - Q &= \sqrt{\frac{p'}{p}}(\Lambda_+ + \rho) - \sqrt{\frac{p}{p'}}(\Lambda_- + \rho), \\ Q^2 &= \frac{1}{pp'}\rho^2 = \frac{1}{12pp'}N(N^2-1), \end{aligned} \tag{3.43}$$

we see that indeed

$$h_{v-Q} = h_{(\Lambda_+,\Lambda_-)}. \tag{3.44}$$

We will denote by $\mathcal{O}_v$ a primary of the W_N algebra and by V_v the corresponding free *chiral* boson vertex operator $e^{iv\cdot X_L}$. On a genus g Riemann surface, correlators of the linear dilaton CFT are nontrivial only if the total charge is $(2-2g)Q$. For instance, the non-vanishing sphere two-point functions must involve a pair of operators V_v and V_{2Q-v}, of equal conformal weights and total charge $2Q$. On the other hand, the fusion rule in the W_N minimal model is such that the correlation function $\langle \mathcal{O}_{v_1}\cdots\mathcal{O}_{v_n}\rangle$ is nonvanishing only if $\sum_{i=1}^n v_i \in \Gamma_{\frac{p'}{p}} + \Gamma_{\frac{p}{p'}} = \Gamma_{\frac{1}{pp'}}$.

For each simple root α_i, we have

$$\sqrt{\frac{p}{p'}}\alpha_i \in \Gamma_{\frac{p}{p'}}, \qquad \sqrt{\frac{p'}{p}}\alpha_i \in \Gamma_{\frac{p'}{p}}. \tag{3.45}$$

The vertex operators

$$V_i^+ = V_{\sqrt{\frac{p}{p'}}\alpha_i}, \qquad V_i^- = V_{-\sqrt{\frac{p'}{p}}\alpha_i}. \tag{3.46}$$

have conformal weight 1, and can be used as screening operators. By inserting screening charges, the contour integrals of these screening operators, we can obtain all correlators of W_N primaries that obey the fusion rule. We can also absorb the background charge with screening charges. This relies on the fact

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha, \tag{3.47}$$

where Δ_+ is the set of all positive roots. So we can write

$$2Q = 4v_0\rho = \sum_{\alpha\in\Delta_+}\left(\sqrt{\frac{p}{p'}}\alpha - \sqrt{\frac{p'}{p}}\alpha\right). \tag{3.48}$$

which may be further written as a sum of non-negative integer multiples of $\sqrt{\frac{p}{p'}}\alpha_i$ and $-\sqrt{\frac{p'}{p}}\alpha_i$, which are the screening operators.

As an example, consider $\mathcal{O}_v$ and its charge conjugate operator $\mathcal{O}_{\overline{v}}$. If V_v is the Coulomb gas representation of $\mathcal{O}_v$, then V_{2Q-v} has the correct dimension and charge (modulo root lattice) to represent $\mathcal{O}_{\overline{v}}$. Alternatively, one may take $V_{\overline{v}}$, which differs from V_{2Q-v} by some screening charges. There is a Weyl reflection w_0 (the longest Weyl group element) such that

$$w_0(\overline{v}) = -v, \qquad w_0(\rho) = -\rho. \tag{3.49}$$

The shifted Weyl transformation by w_0 acts as

$$\begin{aligned} w_0\cdot\overline{v} &= \sqrt{\frac{p'}{p}}(w_0(\overline{\Lambda}_+ + \rho) - \rho) - \sqrt{\frac{p}{p'}}(w_0(\overline{\Lambda}_- + \rho) - \rho) \\ &= 2Q - v. \end{aligned} \tag{3.50}$$

So indeed $\overline{v}$ and $2Q-v$ are identified by Weyl reflection and represent the same W_N primary.

3.4 Sphere three-point function

On the sphere, W_N conformal blocks can also be computed directly from affine Toda theory, by taking the residue of affine Toda conformal blocks as the vertex operators approach those of the W_N minimal model [43]. This spares us the messy screening integrals in the Coulomb gas approach, and allows for easy extraction of explicit three-point functions. Our computation closely follows that of [43].

3.4.1 Two point function and normalization

The two and three point functions in W_N minimal model can be obtained from those of the affine Toda theory, as follows. The affine Toda theory is given by the $N-1$ bosons with linear dilaton described in the previous section, with an additional potential

$$\mu \sum_{i=1}^{N-1} e^{b\alpha_i \cdot X} \tag{3.51}$$

added to the Lagrangian. Following the convention of [43], the background charge $\mathcal{Q}$ is related to b by $\mathcal{Q} = (b + b^{-1})\rho$, where ρ is the Weyl vector. Note that $\mathcal{Q}$ will be related to Q in the previous section by $\mathcal{Q} = iQ$. Normally, one considers the affine Toda theory with real b and $\mathcal{Q}$. To obtain correlators of W_N minimal model, analytic continuation on b as well as a residue procedure will be applied, as we will describe later.

The primary operators in the affine Toda theory are given by

$$V_{\mathbf{v}} = e^{\mathbf{v}\cdot X}. \tag{3.52}$$

$V_{\mathbf{v}}$ and $V_{w\cdot\mathbf{v}}$ represent the same operator (recall that $w \cdot \mathbf{v}$ is the shift Weyl transformation of $\mathbf{v}$ by $w \in W$), but generally come with different normalizations. They are related by

$$V_{\mathbf{v}} = R_w(\mathbf{v}) V_{w\cdot\mathbf{v}}, \tag{3.53}$$

where $R_w(\mathbf{v})$ is the reflection amplitude computed in [45]:

$$R_w(\mathbf{v}) = \frac{\mathbf{A}(w \cdot \mathbf{v})}{\mathbf{A}(\mathbf{v})} = \frac{\mathbf{A}(w(\mathbf{v} - \mathcal{Q}) + \mathcal{Q})}{\mathbf{A}(\mathbf{v})}, \tag{3.54}$$

and

$$\begin{aligned} \mathbf{A}(\mathbf{v}) &= \left[\pi\mu\gamma(b^2)\right]^{\frac{(\mathbf{v}-\mathcal{Q},\rho)}{b}} \prod_{i>j} \Gamma(1 - b(\mathbf{v} - \mathcal{Q}, \mathbf{h}_j - \mathbf{h}_i))\Gamma(-b^{-1}(\mathbf{v} - \mathcal{Q}, \mathbf{h}_j - \mathbf{h}_i)) \\ &= \left[\pi\mu\frac{-1}{\gamma(-b^2)}b^{-4}\right]^{\frac{(\mathbf{v}-\mathcal{Q},\rho)}{b}} \prod_{i>j} \Gamma(1 - b\mathbf{P}_{ij})\Gamma(-b^{-1}\mathbf{P}_{ij}), \end{aligned} \tag{3.55}$$

where $\mathbf{P}_{ij} \equiv (\mathcal{Q} - \mathbf{v}) \cdot (\mathbf{h}_i - \mathbf{h}_j)$. In particular, applying this for the longest Weyl group element w_0, we obtain the relation

$$V_{\bar{\mathbf{v}}} = \frac{\mathbf{A}(2\mathcal{Q} - \mathbf{v})}{\mathbf{A}(\mathbf{v})} V_{2\mathcal{Q}-\mathbf{v}}, \tag{3.56}$$

where $\bar{\mathbf{v}}$ is the conjugate of $\mathbf{v}$. Notice that the function $\mathbf{A}(\mathbf{v})$ has the property $\mathbf{A}(\mathbf{v}) = \mathbf{A}(\bar{\mathbf{v}})$. The operators $V_\mathbf{v}$ are such that the two point function between $V_\mathbf{v}$ and $V_{2\mathcal{Q}-\mathbf{v}}$ is canonically normalized,

$$\langle V_\mathbf{v}(x) V_{2\mathcal{Q}-\mathbf{v}}(0)\rangle = \frac{1}{|x|^{2\Delta_\mathbf{v}}}. \tag{3.57}$$

It follows that that two point function of $V_\mathbf{v}$ and its charge conjugate is

$$\langle V_\mathbf{v}(x) V_{\bar{\mathbf{v}}}(0)\rangle = \frac{\mathbf{A}(2\mathcal{Q} - \mathbf{v})}{\mathbf{A}(\mathbf{v})} \frac{1}{|x|^{2\Delta_\mathbf{v}}}. \tag{3.58}$$

In the W_N minimal model, by (3.74), we have a similar relation (by a slight abuse of notation, we now denote by V_v the primary operator in the W_N minimal model that descends from the corresponding exponential operator in the free boson theory)

$$V_v = R_w(v) V_{w \cdot v}, \tag{3.59}$$

where

$$R_w(v) = \frac{A(w\cdot v)}{A(v)} = \frac{A(w(v-Q)+Q)}{A(v)}, \tag{3.60}$$

and

$$A(v) = \left[\pi\mu\frac{-1}{\gamma(\frac{p'}{p})}\left(\frac{p}{p'}\right)^2\right]^{-\sqrt{\frac{p}{p'}}(v-Q,\rho)} \prod_{i>j}\Gamma(1+\sqrt{\frac{p'}{p}}P_{ij})\Gamma(-\sqrt{\frac{p}{p'}}P_{ij}), \tag{3.61}$$

where $P_{ij} = (v-Q)\cdot(\mathbf{h}_i - \mathbf{h}_j)$. The two point function between V_v and its charge conjugate is then

$$\langle V_v(x)V_{\bar{v}}(0)\rangle^{\text{unnorm}} = \frac{A(2Q-v)}{A(v)}\frac{1}{|x|^{2\Delta_v}}. \tag{3.62}$$

In computing this in the Coulomb gas formalism, appropriated screening charges are inserted, to saturate the background charge. Consequently, the vacuum isn't canonically normalized. In fact, we have

$$\langle 1\rangle^{\text{unnorm}} = \frac{A(2Q)}{A(0)}. \tag{3.63}$$

The normalized correlators are related by

$$\langle V_1\cdots V_n\rangle = \frac{\langle V_1\cdots V_n\rangle^{\text{unnorm}}}{\langle 1\rangle^{\text{unnorm}}} = \frac{A(0)}{A(2Q)}\langle V_1\cdots V_n\rangle^{\text{unnorm}}. \tag{3.64}$$

Here again the "unnormalized" n-point function is understood to be computed with appropriated screening charges inserted. Next, we define the normalized operators $\widetilde{V}_v$ by

$$\widetilde{V}_v = \sqrt{\frac{A(v)A(2Q)}{A(2Q-v)A(0)}}V_v \equiv B(v)V_v, \tag{3.65}$$

and then we have

$$\left\langle \widetilde{V}_v(x)\widetilde{V}_{\bar{v}}(0)\right\rangle = \frac{1}{|x|^{2\Delta_v}}. \tag{3.66}$$

3.4.2 Extracting correlation functions from affine Toda theory

Let us proceed to the three point functions in the W_N minimal model:

$$\langle V_{v_1} V_{v_2} V_{v_3} \rangle^{\text{unnorm}} = \frac{C_{W_N}(v_1, v_2, v_3)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}. \tag{3.67}$$

where Δ_i denotes the total scaling dimension of V_{v_i}. The normalized three point functions of the normalized operators $\widetilde{V}_{v_i}$ are given by

$$\left\langle \widetilde{V}_{v_1} \widetilde{V}_{v_2} \widetilde{V}_{v_3} \right\rangle = B(v_1)\, B(v_2)\, B(v_3)^{-1} \langle V_{v_1} V_{v_2} V_{2Q-\bar{v}_3} \rangle^{\text{unnorm}}, \tag{3.68}$$

and the structure constants, with two-point functions normalized to unity, are

$$C_{nor}(v_1, v_2, v_3) = B(v_1)\, B(v_2)\, B(v_3)^{-1}\, C_{W_N}(v_1, v_2, 2Q - \bar{v}_3). \tag{3.69}$$

Nontrivial data are contained in the structure constants $C_{W_N}(v_1, v_2, v_3)$, which we now compute.

In the affine Toda theory, the three point functions of the operators (3.52) are of the form

$$\langle V_{\mathbf{v}_1} V_{\mathbf{v}_2} V_{\mathbf{v}_3} \rangle = \frac{C_{\text{Toda}}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}. \tag{3.70}$$

The structure constants $C_{\text{Toda}}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ are computed in [43]. They have poles when the relation

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + b \sum_{k=1}^{N-1} s_k \alpha_k + \frac{1}{b} \sum_{k=1}^{N-1} s'_k \alpha_k = 2\mathcal{Q} \tag{3.71}$$

is obeyed, where s_k and s'_k are nonnegative integers. The pole structure is as follows. For general $\mathbf{v}_i$'s, define a charge vector $\epsilon = \sum \epsilon_i \alpha_i$ through the following equation

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + b \sum_{k=1}^{N-1} s_k \alpha_k + \frac{1}{b} \sum_{k=1}^{N-1} s'_k \alpha_k + \epsilon = 2\mathcal{Q}. \tag{3.72}$$

The relation (3.71) is obeyed when $\epsilon_i = 0$, $i = 1, \cdots, N-1$. This is an order $N-1$ pole of the structure constant $C_{\text{Toda}}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, understood as a function of ϵ. The W_N minimal model structure constant, $C_{W_N}(v_1, v_2, v_3)$, is computed by taking $N-1$ successive residues,[9]

$$\text{res}_{\epsilon_1 \to 0} \text{res}_{\epsilon_2 \to \epsilon_1} \cdots \text{res}_{\epsilon_{N-1} \to \epsilon_{N-2}} C_{\text{Toda}}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), \tag{3.73}$$

and then analytically continuing to the following imaginary values of b and $\mathbf{v}_i$,

$$b = -i\sqrt{\frac{p'}{p}}, \qquad \mathbf{v}_j = i v_j. \tag{3.74}$$

The relation (3.71) is always satisfied by the v_i's obeying the W_N fusion rules in some Weyl chamber. The overall normalization of the three point function can be then fixed by requiring

$$C_{W_N}(0, 0, 2Q) = 1. \tag{3.75}$$

In [43], by bootstrapping the sphere four point function, the following class of three point function coefficients were computed in the affine Toda theory:

$$\begin{aligned} &C_{\text{Toda}}(\mathbf{v}_1, \mathbf{v}_2, \varkappa\omega^{N-1}) \\ &= \left[\pi\mu\gamma(b^2) b^{2-2b^2}\right]^{\frac{(2\mathcal{Q} - \sum \mathbf{v}_i, \rho)}{b}} \frac{(\Upsilon(b))^N \, \Upsilon(\varkappa) \prod_{\alpha \in \Delta_+} \Upsilon\Big((\mathcal{Q} - \mathbf{v}_1)\cdot\alpha\Big) \Upsilon\Big((\mathcal{Q} - \mathbf{v}_2)\cdot\alpha\Big)}{\prod_{i,j=1}^{N-1} \Upsilon\Big(\frac{\varkappa}{N} + (\mathbf{v}_1 - \mathcal{Q})\cdot\mathbf{h}_i + (\mathbf{v}_2 - \mathcal{Q})\cdot\mathbf{h}_j\Big)}, \end{aligned} \tag{3.76}$$

where $\varkappa$ is a real number, ω^{N-1} is the fundamental weight vector associated to the anti-fundamental representation, and the $\mathbf{h}_k$'s are charge vectors defined as

$$\mathbf{h}_k = \omega^1 - \sum_{i=1}^{k-1} \alpha_i, \tag{3.77}$$

[9] The residue (3.73) can also be computed using a Coulomb gas integral. See (1.24) of [43].

where ω^1 is the first fundamental weight, associated with the fundamental representation. The function Υ is defined by

$$\log\Upsilon(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{\mathcal{Q}}{2}-x\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{\mathcal{Q}}{2}-x\right)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]. \tag{3.78}$$

It obeys the identities,

$$\begin{aligned}\Upsilon(x+b) &= \gamma(bx)b^{1-2bx}\Upsilon(x),\\ \Upsilon(x+1/b) &= \gamma(x/b)b^{2x/b-1}\Upsilon(x),\\ \Upsilon(x) &= \Upsilon(b+1/b-x),\end{aligned} \tag{3.79}$$

and has zeros at $x=-nb-m/b$ and at $x=(1+n)b+(1+m)/b$, for nonnegative integers n,m.

The procedure of computing $C_{W_N}(v_1,v_2,v_3)$ from the residue of (3.76), when v_3 is proportional to ω^{N-1}, is carried out in Appendix 3.A. The result is

$$\begin{aligned}&C_{W_N}\left(v_1,v_2,\left(\sqrt{\frac{p'}{p}}n-\sqrt{\frac{p}{p'}}m\right)\omega_{N-1}\right)\\ &=\left(\frac{p'}{p}\right)^{\sum_{j=1}^{N-2}(s_js'_{j+1}-s_{j+1}s'_j)}\left[\frac{-\mu\pi}{\gamma(\frac{p'}{p})}\right]^{\sum\limits_{k=1}^{N-1}s_k}\left[\frac{-\mu'\pi}{\gamma(\frac{p}{p'})}\right]^{\sum\limits_{k=1}^{N-1}s'_k}\left(\prod_{k=0}^{s'_{N-1}-1}\prod_{l=0}^{s_{N-1}-1}\frac{-1}{\left(\sqrt{\frac{p'}{p}}(n-l)-\sqrt{\frac{p}{p'}}(m-k)\right)^2}\right)\\ &\quad\times\left[\prod_{l=0}^{s_{N-1}-1}\gamma(1+m-\frac{p'}{p}(n-l))\right]\left[\prod_{k=0}^{s'_{N-1}-1}\gamma(1+n-\frac{p}{p'}(m-k))\right]\prod_{j=1}^{N-1}R_{j,0}^{s_{j,j-1},s'_{j,j-1}},\end{aligned} \tag{3.80}$$

where $R_{j,0}^{s_{j,j-1},s'_{j,j-1}}$ is the $\epsilon = 0$ value of

$$R_{j,\epsilon}^{s_{j,j-1},s'_{j,j-1}} = \left(\prod_{k=1}^{s'_{j,j-1}} \prod_{l=1}^{s_{j,j-1}} \frac{-1}{(\epsilon \cdot \mathbf{h}_j + \sqrt{\frac{p}{p'}}k - \sqrt{\frac{p'}{p}}l)^2} \prod_{i=j+1}^{N} \frac{1}{(P_{ij}^1 - \sqrt{\frac{p}{p'}}k + \sqrt{\frac{p'}{p}}l)^2} \frac{1}{(P_{ij}^2 - \sqrt{\frac{p}{p'}}k + \sqrt{\frac{p'}{p}}l)^2} \right)$$
$$\times \left[\prod_{l=1}^{s_{j,j-1}} \gamma(\epsilon \cdot \mathbf{h}_j + \frac{p'}{p}l) \prod_{i=j+1}^{N} \gamma(\sqrt{\frac{p'}{p}} P_{ij}^1 + \frac{p'}{p}l) \gamma(\sqrt{\frac{p'}{p}} P_{ij}^2 + \frac{p'}{p}l) \right]$$
$$\times \left[\prod_{k=1}^{s'_{j,j-1}} \gamma(\epsilon \cdot \mathbf{h}_j + \frac{p}{p'}k) \prod_{i=j+1}^{N} \gamma(-\sqrt{\frac{p}{p'}} P_{ij}^1 + \frac{p}{p'}k) \gamma(-\sqrt{\frac{p}{p'}} P_{ij}^2 + \frac{p}{p'}k) \right]. \tag{3.81}$$

P_{ij}^1 and P_{ij}^2 are defined as $P_{ij}^r = (v_r - Q) \cdot (\mathbf{h}_i - \mathbf{h}_j)$, $r = 1, 2$, and the function $\gamma(x)$ is defined as $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. μ' is the dual cosmological constant, which is related to the cosmological constant μ by

$$\mu' = \frac{1}{\pi \gamma \left(-\frac{p}{p'} \right)} \left[\pi \mu \gamma \left(-\frac{p'}{p} \right) \right]^{-\frac{p}{p'}}. \tag{3.82}$$

In the special case of $s'_i = 0$ for all $i = 1, \cdots, N-1$, the expressions simplify:

$$C_{W_N} \left(v_1, v_2, \left(\sqrt{\frac{p'}{p}} n - \sqrt{\frac{p}{p'}} m \right) \omega_{N-1} \right)$$
$$= \left[\frac{-\mu\pi}{\gamma(\frac{p'}{p})} \right]^{\sum_{k=1}^{N-1} s_k} \left[\prod_{l=0}^{s_{N-1}-1} \gamma(1 + m - \frac{p'}{p}(n-l)) \right] \prod_{j=1}^{N-1} R_{j,0}^{s_{j,j-1},0}, \tag{3.83}$$

and

$$R_{j,0}^{s_{j,j-1},0} = \left[\prod_{l=1}^{s_{j,j-1}} \gamma(\frac{p'}{p}l) \prod_{i=j+1}^{N} \gamma(\sqrt{\frac{p'}{p}} P_{ij}^1 + \frac{p'}{p}l) \gamma(\sqrt{\frac{p'}{p}} P_{ij}^2 + \frac{p'}{p}l) \right]. \tag{3.84}$$

3.4.3 Large N factorization

In this section, we compute three point functions of W_N primaries $(\mathbf{f}, 0)$, $(\mathbf{f}, \mathbf{f})$, and/or their charge conjugates, with the primary (Λ_+, Λ_-) where $\Lambda_\pm$ are the symmetric or anti-symmetric tensor products of $\mathbf{f}$ or $\overline{\mathbf{f}}$. While the former are thought to be dual to elementary scalar fields in the bulk AdS_3 theory, the latter are expected to be composite particles, or bound states, of the former. If this interpretation is correct, then the three point functions in the large N limit must factorize into products of two-point functions, as the bound states become unbound at zero bulk coupling. We will see that this is indeed the case. Our method can be carried out more generally to identify all elementary particles and their bound states in the bulk at large N, including the light states.

Massive scalars and their bound states

To begin with, let us consider the three point function of $(\bar{\mathbf{f}}, 0)$, $(\bar{\mathbf{f}}, 0)$, and $(A, 0)$, where A is the antisymmetric tensor product of two $\mathbf{f}$'s. Note that in the large N limit, $(\mathbf{f}, 0)$ has scaling dimension $\Delta_{(\mathbf{f},0)} = 1 + \lambda$, while $(A, 0)$ has twice the dimension, and is expected to be the lowest bound state of two $(\mathbf{f}, 0)$'s. The charge vectors are

$$v_1 = v_2 = \sqrt{\frac{p'}{p}}\omega_{N-1}, \qquad v_3 = \sqrt{\frac{p'}{p}}\omega_2. \tag{3.85}$$

The structure constant, extracted using affine Toda theory, is

$$C_{W_N}\left(\sqrt{\frac{p'}{p}}\omega_{N-1}, \sqrt{\frac{p'}{p}}\omega_{N-1}, 2Q - \sqrt{\frac{p'}{p}}\omega_{N-2}\right) = \left[\frac{-\mu\pi}{\gamma(\frac{p'}{p})}\right]\gamma\left(1 - \frac{p'}{p}\right)\gamma\left(2\frac{p'}{p} - 1\right). \tag{3.86}$$

By (3.69), the normalized structure constant are computed to be

$$\begin{aligned} C_{nor} &= \sqrt{2}\left[-\frac{(1-\frac{1}{N})\Gamma(-\lambda)\Gamma(\frac{2\lambda}{N})\Gamma(\lambda - \frac{\lambda}{N})\Gamma(-1-\frac{\lambda}{N})}{(1+\frac{\lambda}{N})^3\Gamma(\lambda)\Gamma(-\lambda+\frac{\lambda}{N})\Gamma(-\frac{2\lambda}{N})\Gamma(\frac{\lambda}{N})}\right]^{\frac{1}{2}} \\ &= \sqrt{2} - \frac{2 + 4\lambda + \pi\lambda\cot\pi\lambda + 2\lambda(\gamma + \psi(\lambda))}{\sqrt{2}N} + \mathcal{O}(\frac{1}{N^2}), \end{aligned} \tag{3.87}$$

where γ is the Euler-Mascheroni constant, and the $\psi(\lambda)$ is the digamma function.

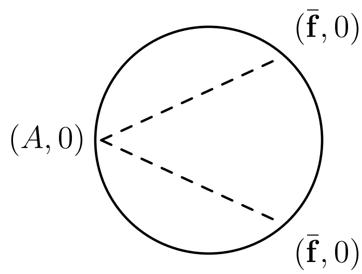
In the infinite N limit, the bulk theory is expected to become free. If we denote $(\mathbf{f}, 0)$ by ϕ, the OPE of ϕ should behave like that of a free field of dimension $\Delta_{(\mathbf{f},0)}$. Given the two-point function

$$\left\langle \phi(x)\bar{\phi}(0)\right\rangle = \frac{1}{|x|^{2\Delta_{(\mathbf{f},0)}}}, \tag{3.88}$$

the product of two ϕ's, normalized as $\frac{1}{\sqrt{2}}\phi^2$, has the two point function $1/|x|^{4\Delta_{(\mathbf{f},0)}}$. With the identification

$$(A,0) \sim \frac{1}{\sqrt{2}}\phi^2, \tag{3.89}$$

i.e. $(A,0)$ as a bound state of two ϕ's that becomes free in the large N limit, the three-point function coefficient is indeed $\sqrt{2}$, agreeing with the free correlator $\langle\bar{\phi}(x_1)\bar{\phi}(x_2)\,\frac{1}{\sqrt{2}} : \phi^2(x_3) :\rangle$.



The next example we consider is the three point function of two $(\bar{\mathbf{f}},0)$'s and $(S,0)$, where S is the symmetric tensor product of two $\mathbf{f}$'s. In the large N limit, $(S,0)$ has dimension $2\Delta_{(\mathbf{f},0)}+2$, and may be expected to be an excited resonance of two $(\mathbf{f},0)$'s. The charge vectors of the three primaries are

$$v_1 = v_2 = \sqrt{\frac{p'}{p}}\omega_{N-1}, \qquad v_3 = \sqrt{\frac{p'}{p}}\,2\omega_1. \tag{3.90}$$

The structure constant computed from Coulomb integral is very simple:

$$C_{W_N}\left(\sqrt{\frac{p'}{p}}\omega_{N-1}, \sqrt{\frac{p'}{p}}\omega_{N-1}, 2Q - \sqrt{\frac{p'}{p}}2\omega_{N-1}\right) = 1, \tag{3.91}$$

and the normalized structure constant is

$$\begin{aligned} C_{nor} &= \left[\frac{2\Gamma(-\lambda)\Gamma(\frac{\lambda}{N})\Gamma(-2-\frac{2\lambda}{N})\Gamma(2+\lambda+\frac{\lambda}{N})}{N\Gamma(\lambda)\Gamma(-1-\frac{\lambda}{N})\Gamma(2+\frac{2\lambda}{N})\Gamma(-1-\lambda-\frac{\lambda}{N})}\right]^{\frac{1}{2}} \\ &= \frac{1+\lambda}{\sqrt{2}} + \frac{\lambda(1+\lambda)(-4+2\gamma+\psi(-1-\lambda)+\psi(2+\lambda))}{2\sqrt{2}N} + \mathcal{O}(\frac{1}{N^2}). \end{aligned} \tag{3.92}$$

Let us compare $(S,0)$ with the primary that appears in the OPE of two free fields ϕ's at level $(1,1)$, with normalized two-point function,

$$\frac{1}{\sqrt{2}\Delta_{(\mathbf{f},0)}}(\phi\partial\bar{\partial}\phi-\partial\phi\bar{\partial}\phi). \tag{3.93}$$

The structure constant of (3.93) with two $\bar{\phi}$'s is $\Delta_{(\mathbf{f},0)}/\sqrt{2}$, precisely agreeing with (3.92) in the large N limit, as $\Delta_{(\mathbf{f},0)}=1+\lambda$. This leads us to identify

$$(S,0)\sim\frac{1}{\sqrt{2}\Delta_{(\mathbf{f},0)}}(\phi\partial\bar{\partial}\phi-\partial\phi\bar{\partial}\phi). \tag{3.94}$$

Next, we consider the three point function of $(\mathbf{f},0)$, $(\bar{\mathbf{f}},0)$, and $(adj,0)$, where adj is the adjoint representation of $SU(N)$. A similar computation gives[10]

$$\begin{aligned}C_{nor}((\mathbf{f},0),(\bar{\mathbf{f}},0),(adj,0))&=\left[\frac{(1-\frac{1}{N})\Gamma(-\lambda)\Gamma(\lambda-\frac{\lambda}{N})}{(1+\frac{\lambda}{N})^2\Gamma(\lambda)\Gamma(-\lambda+\frac{\lambda}{N})}\right]^{\frac{1}{2}}\\&=1-\frac{1+\lambda+\frac{1}{2}\pi\lambda\cot\pi\lambda-\lambda\psi(\lambda)}{N}+\mathcal{O}(\frac{1}{N^2}).\end{aligned} \tag{3.95}$$

This allows us to identify

$$(adj,0)\sim\phi\bar{\phi}, \tag{3.96}$$

in large N limit.

As a simple check of our identification, we can compute the three point function of $(A,0)$, $(\overline{S},0)$, and $(adj,0)$, which is expected to factorize into three two-point functions (i.e. $\sim\langle\phi\bar{\phi}\rangle^3$) in the large N limit. Indeed, with the three charge vectors

$$v_1=\sqrt{\frac{p'}{p}}\omega_2,\qquad v_2=\sqrt{\frac{p'}{p}}2\omega_{N-1},\qquad v_3=\sqrt{\frac{p'}{p}}(\omega_1+\omega_{N-1}), \tag{3.97}$$

[10] Here and from now on, we write $C_{nor}(v_1,v_2,v_3)$ in terms of the three pairs of representations rather than charge vectors.

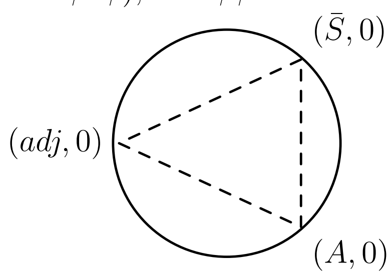
we have

$$C_{W_N}\left(\sqrt{\frac{p'}{p}}\omega_2, \sqrt{\frac{p'}{p}}2\omega_{N-1}, 2Q-\sqrt{\frac{p'}{p}}(\omega_1+\omega_{N-1})\right)$$
$$=\left[\frac{-\mu\pi}{\gamma(\frac{p'}{p})}\right]^{N-2}\gamma(1-2\frac{p'}{p})\gamma(\frac{p'}{p})\left[\prod_{i=3}^{N}\gamma\left(\left(\frac{p'}{p}-1\right)(2-i)\right)\gamma\left(\frac{p'}{p}(\delta_{i,N}-1+i)+(2-i)\right)\right], \tag{3.98}$$

and for the normalized structure constant,

$$\begin{aligned}C_{nor}((A,0),(\overline{S},0),(adj,0)) &= \left[\frac{N^4(1+\lambda)^3\Gamma(1+\lambda)\Gamma\left(-1+\lambda+\frac{\lambda}{N}\right)}{(N+\lambda)^2(N+2\lambda)^2\Gamma(-1-\lambda)\Gamma\left(2+\lambda+\frac{\lambda}{N}\right)}\right]^{\frac{1}{2}} \\ &= (1+\lambda)-\frac{\lambda(1+\lambda)(6+\psi(-1-\lambda)+\psi(2+\lambda))}{2N}+\mathcal{O}(\frac{1}{N^2}),\end{aligned} \tag{3.99}$$

which is indeed reproduced in the large N limit by the three point function of free field products $\frac{1}{\sqrt{2}}\phi\phi$, $\frac{1}{\sqrt{2}\Delta_{(\mathbf{f},0)}}(\bar{\phi}\partial\bar{\partial}\bar{\phi}-\partial\bar{\phi}\bar{\partial}\bar{\phi})$, and $\phi\bar{\phi}$.



Light states

The bound states of basic primaries discussed so far can be easily guessed by comparison the scaling dimensions in the large N limit. This is less obvious with the light states, which are labeled by a pair of identical representations, i.e. of the form (R,R).

To begin with, consider the light state $(\mathbf{f},\mathbf{f})$, whose dimension in the large N limit is $\Delta_{(\mathbf{f},\mathbf{f})}=\lambda^2/N$. The OPE of two $(\mathbf{f},\mathbf{f})$'s contains (A,A) and (S,S), whose dimensions in the large N limit are both $2\Delta_{(\mathbf{f},\mathbf{f})}$, as well as (A,S) and (S,A), whose dimensions are $2\Delta_{(\mathbf{f},\mathbf{f})}+2$. A linear combination of (A,A) and (S,S) is thus expected to be the lowest bound state of two $(\mathbf{f},\mathbf{f})$'s. This linear combination can be determined by inspecting the three-point functions of two $(\bar{\mathbf{f}},\bar{\mathbf{f}})$'s with (A,A) and (S,S).

The normalized structure constant of two $(\bar{\mathbf{f}},\bar{\mathbf{f}})$'s with (A,A) is computed to be

$$\begin{aligned} C_{nor}((\bar{\mathbf{f}},\bar{\mathbf{f}}),(\bar{\mathbf{f}},\bar{\mathbf{f}}),(A,A)) = &\left[\frac{(N+\lambda)\Gamma\left(1-\lambda\right)\Gamma\left(\frac{2\lambda}{N}\right)\Gamma\left(\frac{3\lambda}{N}\right)^2\Gamma\left(\frac{-3\lambda}{N+\lambda}\right)^2\Gamma\left(\frac{-2\lambda}{N+\lambda}\right)}{N\Gamma\left(\frac{-3\lambda}{N}\right)^2\Gamma\left(\frac{-2\lambda}{N}\right)\Gamma\left(1+\lambda\right)\Gamma\left(\frac{-N}{N+\lambda}\right)\Gamma\left(\frac{2\lambda}{N+\lambda}\right)}\right. \\ &\left.\times\frac{\Gamma\left(\frac{-\lambda}{N+\lambda}\right)\Gamma\left(\frac{N\lambda}{N+\lambda}\right)\Gamma\left(\frac{N+\lambda}{N}\right)\Gamma\left(1+\lambda-\frac{\lambda}{N}\right)\Gamma\left(\frac{N+2\lambda-N\lambda}{N+\lambda}\right)}{\Gamma\left(\frac{3\lambda}{N+\lambda}\right)^2\Gamma\left(\frac{-N\lambda}{N+\lambda}\right)\Gamma\left(\frac{N(1+\lambda)}{N+\lambda}\right)\Gamma\left(\frac{N+\lambda-N\lambda}{N}\right)\Gamma\left(\frac{N-\lambda}{N}\right)}\right]^{\frac{1}{2}} \\ = 1 + &\frac{\lambda^2(-\pi\cot\pi\lambda+\pi^2\lambda\cot^2\pi\lambda-18\gamma-2\psi(\lambda)-2\lambda\psi^{(1)}(\lambda))}{2N^2}+\mathcal{O}(\frac{1}{N^3}), \end{aligned} \tag{3.100}$$

and with (S,S),

$$\begin{aligned} &C_{nor}((\bar{\mathbf{f}},\bar{\mathbf{f}}),(\bar{\mathbf{f}},\bar{\mathbf{f}}),(S,S)) \\ &= \left[\frac{2^{-\frac{4\lambda^2}{N^2+N\lambda}}(N+1)^2\Gamma\left(1-\lambda\right)\Gamma\left(\frac{\lambda+N\lambda}{N}\right)\Gamma\left(\frac{-\lambda-N\lambda}{N+\lambda}\right)\Gamma\left(\frac{N+3\lambda}{2N+2\lambda}\right)\Gamma\left(\frac{1}{2}-\frac{\lambda}{N}\right)\Gamma\left(\frac{N+\lambda+N\lambda}{N+\lambda}\right)}{N^2\Gamma\left(\lambda\right)\Gamma\left(\frac{N-\lambda}{2(N+\lambda)}\right)\Gamma\left(\frac{-N\lambda}{N+\lambda}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{N}\right)\Gamma\left(\frac{N+2\lambda+N\lambda}{N+\lambda}\right)\Gamma\left(\frac{N-\lambda-N\lambda}{N}\right)}\right]^{\frac{1}{2}} \\ &= 1+\frac{\lambda^2(\pi\cot\pi\lambda-\pi^2\lambda\csc^2\pi\lambda+2(\gamma+\psi(\lambda)+\lambda\psi^{(1)}(\lambda)))}{2N^2}+\mathcal{O}(\frac{1}{N^3}), \end{aligned} \tag{3.101}$$

where $\psi^{(1)}(\lambda)$ is the trigamma function. We will denote the operator $(\mathbf{f},\mathbf{f})$ by ω, and the lowest nontrivial operator in the OPE of two such light operators by ω^2. Anticipating large N factorization, if ω were a free field, then the product operator with correctly normalized two-point function is $\frac{1}{\sqrt{2}}\omega^2$. The structure constant fusing two ω's into their bound state $\frac{1}{\sqrt{2}}\omega^2$ is therefore $\sqrt{2}$ in the free limit. This is indeed the case: the three point function coefficient of two $(\bar{\mathbf{f}},\bar{\mathbf{f}})$'s and the linear combination $\frac{1}{\sqrt{2}}((S,S)+(A,A))$ is

$$C_{nor}=\sqrt{2}-\frac{4\sqrt{2}\gamma\lambda^2}{N^2}+\mathcal{O}(\frac{1}{N^3}). \tag{3.102}$$

This leads to the identification

$$\frac{(S,S)+(A,A)}{\sqrt{2}}\sim\frac{1}{\sqrt{2}}\omega^2. \tag{3.103}$$

The other linear combination

$$\frac{(S,S)-(A,A)}{\sqrt{2}} \tag{3.104}$$

is orthogonal to ω^2 and has vanishing three point function with two $(\bar{\mathbf{f}},\bar{\mathbf{f}})$'s in the large N limit. It is therefore a new elementary light particle.

To identify the first excited composite state of two $(\mathbf{f},\mathbf{f})$'s as a linear combination of (A,S) with (S,A), we compute the structure constants

$$\begin{aligned}C_{nor}((\bar{\mathbf{f}},\bar{\mathbf{f}}),(\bar{\mathbf{f}},\bar{\mathbf{f}}),(A,S)) &= \left[\frac{\pi^2(N-1)(N+\lambda)^6\csc\frac{2\pi\lambda}{N}\csc\frac{N\pi\lambda}{N+\lambda}\Gamma(1-\lambda)\Gamma\left(\frac{N}{N+\lambda}\right)\Gamma\left(\frac{-N-\lambda}{N}\right)}{N^6\Gamma(\lambda)\Gamma\left(\frac{\lambda}{N}\right)\Gamma\left(\frac{-N}{N+\lambda}\right)\Gamma\left(\frac{-2\lambda}{N+\lambda}\right)\Gamma\left(\frac{-N\lambda}{N+\lambda}\right)^2\Gamma\left(\frac{(1+N)\lambda}{N+\lambda}\right)}\right.\\ &\quad\left.\times\frac{\Gamma\left(\frac{N+3\lambda}{N+\lambda}\right)\Gamma\left(\frac{N-\lambda+N\lambda}{N}\right)\Gamma\left(\frac{N-N\lambda}{N+\lambda}\right)}{\Gamma\left(1-\lambda+\frac{\lambda}{N}\right)\Gamma\left(\frac{3N-2\lambda}{N}\right)^2}\right]^{\frac{1}{2}}\\ &=\frac{\lambda^2}{2N}-\frac{\lambda^2(1-3\lambda+\pi\lambda\cot\pi\lambda+2\lambda\gamma+2\lambda\psi(\lambda))}{2N^3}+\mathcal{O}(\frac{1}{N^4}),\end{aligned} \tag{3.105}$$

and

$$\begin{aligned}C_{nor}((\bar{\mathbf{f}},\bar{\mathbf{f}}),(\bar{\mathbf{f}},\bar{\mathbf{f}}),(S,A)) &= \left[\frac{\pi^2(N-1)N^6\csc\pi\lambda\csc\frac{2N\pi}{N+\lambda}\Gamma\left(\frac{-N}{N+\lambda}\right)\Gamma\left(\frac{N+\lambda}{N}\right)\Gamma\left(1-\frac{2\lambda}{N}\right)}{(N+\lambda)^6\Gamma(\lambda)^2\Gamma\left(\frac{2\lambda}{N}\right)\Gamma\left(\frac{-\lambda-N\lambda}{N}\right)\Gamma\left(\frac{-\lambda}{N+\lambda}\right)\Gamma\left(\frac{-N\lambda}{N+\lambda}\right)}\right.\\ &\quad\left.\times\frac{\Gamma\left(1+\lambda+\frac{\lambda}{N}\right)\Gamma\left(\frac{N+2\lambda-N\lambda}{N+\lambda}\right)\Gamma\left(\frac{N+\lambda+N\lambda}{N+\lambda}\right)}{\Gamma\left(\frac{N(1+\lambda)}{N+\lambda}\right)\Gamma\left(\frac{-N-\lambda}{N}\right)\Gamma\left(\frac{3N+5\lambda}{N+\lambda}\right)^2}\right]^{\frac{1}{2}}\\ &=\frac{\lambda^2}{2N}+\frac{\lambda^2(1-5\lambda+\pi\lambda\cot\pi\lambda+2\lambda\gamma+2\lambda\psi(\lambda))}{2N^3}+\mathcal{O}(\frac{1}{N^4}).\end{aligned} \tag{3.106}$$

Comparing its large N limit with the free field products leads to the identification of $\frac{1}{\sqrt{2}}((A,S)+(S,A))$ as the two-particle state,

$$\frac{(A,S)+(S,A)}{\sqrt{2}}\sim\frac{1}{\sqrt{2}\Delta_{(\mathbf{f},\mathbf{f})}}(\omega\partial\bar{\partial}\omega-\partial\omega\bar{\partial}\omega). \tag{3.107}$$

Note that the RHS of (3.107) has the correctly normalized two-point function provided that the dimension of ω is $\Delta_{(\mathbf{f},\mathbf{f})}=\lambda^2/N$. The orthogonal linear combination $\frac{1}{\sqrt{2}}((A,S)-(S,A))$

has vanishing three point function with two $(\bar{\mathbf{f}}, \bar{\mathbf{f}})$'s at infinite N.

There is an important subtlety, pointed out in [12]: while $\frac{1}{\Delta_{(\mathbf{f},\mathbf{f})}}\partial\bar{\partial}\omega$ is a descendant of ω, it is not truly an elementary particle. In fact, direct inspection of three-point functions at large N shows that it should be identified with the bound state of $\phi = (\mathbf{f}, 0)$ and $\widetilde{\phi} = (0, \mathbf{f})$, i.e.

$$\frac{1}{\Delta_{(\mathbf{f},\mathbf{f})}}\partial\bar{\partial}\omega \sim \phi\,\widetilde{\phi}. \tag{3.108}$$

This is not in conflict with the statement that ω itself is an elementary particle, since in the large N limit $\partial\bar{\partial}\omega$ (without the normalization factor $1/\Delta_{(\mathbf{f},\mathbf{f})}$) becomes null. With the identification (3.108), we can also express (3.107) as

$$\frac{(A,S)+(S,A)}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}\left(\omega\phi\widetilde{\phi} - \frac{1}{\Delta_{(\mathbf{f},\mathbf{f})}}\partial\omega\bar{\partial}\omega\right). \tag{3.109}$$

In the next subsection, we will see a nontrivial consistency check of this identification.

Light states bound to massive scalars

So far we have seen that the massive elementary particles and the light particles interact weakly among themselves at large N. One can also see that the bound state between a massive scalar and a light state becomes free in the large N limit. We will consider the example of $(\mathbf{f}, 0)$ and $(\mathbf{f}, \mathbf{f})$ fusing into $(A, \mathbf{f})$ or $(S, \mathbf{f})$. At infinite N, the operators $(A, \mathbf{f})$ and $(S, \mathbf{f})$ have the same dimension as that of the basic primary $(\mathbf{f}, 0)$, namely $\Delta_{(\mathbf{f},0)} = 1+\lambda$, and the light state $(\mathbf{f}, \mathbf{f})$ has dimension zero. A linear combination of $(A, \mathbf{f})$ and $(S, \mathbf{f})$ should be identified with the lowest bound state of $(\mathbf{f}, 0)$ and $(\bar{\mathbf{f}}, \bar{\mathbf{f}})$. This is seen from the three point function coefficients

$$C_{nor}((\bar{\mathbf{f}},0),(\bar{\mathbf{f}},\bar{\mathbf{f}}),(A,\mathbf{f})) = \left[-\frac{\pi(N-1)^2\csc\frac{2\pi\lambda}{N+\lambda}\csc\frac{N\pi\lambda}{N+\lambda}\sin\frac{N\pi}{N+\lambda}\Gamma\left(\frac{\lambda-N\lambda}{N+\lambda}\right)\Gamma\left(1+\frac{\lambda}{N+\lambda}\right)^2}{N^2\Gamma\left(\frac{-N\lambda}{N+\lambda}\right)^2\Gamma\left(\frac{N(1+\lambda)}{N+\lambda}\right)\Gamma\left(\frac{N+3\lambda}{N+\lambda}\right)^2}\right]^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} + \frac{\lambda}{2\sqrt{2}N}(\pi\cot\pi\lambda + 2\gamma + 2\psi(\lambda)) + \mathcal{O}(\frac{1}{N^2}), \tag{3.110}$$

and

$$C_{nor}((\bar{\mathbf{f}},0),(\bar{\mathbf{f}},\bar{\mathbf{f}}),(S,\mathbf{f})) = \left[\frac{2^{-1+\frac{4\lambda}{N+\lambda}}(N+1)\Gamma\left(\frac{N-N\lambda}{N+\lambda}\right)\Gamma\left(\frac{N+\lambda+N\lambda}{N+\lambda}\right)\Gamma\left(\frac{N+2\lambda}{2(N+\lambda)}\right)}{N\Gamma\left(\frac{N+2\lambda+N\lambda}{N+\lambda}\right)\Gamma\left(\frac{-\lambda}{2(N+\lambda)}\right)\Gamma\left(\frac{N+\lambda-N\lambda}{N+\lambda}\right)}\right]^{\frac{1}{2}} \tag{3.111}$$

$$= \frac{1}{\sqrt{2}} - \frac{\lambda}{2\sqrt{2}N}(\pi\cot\pi\lambda + 2\gamma + 2\psi(\lambda)) + \mathcal{O}(\frac{1}{N^2}).$$

By comparing with the free field product of the elementary massive scalar ϕ with the light field ω, we can identify

$$\frac{(A,\mathbf{f})+(S,\mathbf{f})}{\sqrt{2}} \sim \phi\omega. \tag{3.112}$$

The orthogonal linear combination $\frac{1}{\sqrt{2}}((A,\mathbf{f})-(S,\mathbf{f}))$ has vanishing three point function with $(\mathbf{f},0)$ and $(\mathbf{f},\mathbf{f})$ in the infinite N limit. This is a new elementary particle, with the same mass as that of ϕ in the infinite N limit.[11]

One can further study the fusion of $(0,\mathbf{f})$ with $(A,\mathbf{f})$ into (A,S), and the fusion of $(0,\mathbf{f})$ with $(S,\mathbf{f})$ into (S,A). The normalized structure constants for both three-point functions are $1/\sqrt{2}$ in the infinite N limit. In particular,

$$C_{nor}\left((0,\bar{\mathbf{f}}),\frac{(\bar{A},\bar{\mathbf{f}})+(\bar{S},\bar{\mathbf{f}})}{\sqrt{2}},\frac{(A,S)+(S,A)}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \mathcal{O}(\frac{1}{N}). \tag{3.113}$$

[11]We thank S. Raju for emphasizing this point. Note that on dimensional grounds, if $\frac{1}{\sqrt{2}}((A,\mathbf{f})-(S,\mathbf{f}))$ were a bound state, it could only be that of $(\mathbf{f},0)$ with a light state of the form (R,R), but by fusion rule R must be $\mathbf{f}$, and we already know that $\frac{1}{\sqrt{2}}((A,\mathbf{f})-(S,\mathbf{f}))$ is orthogonal to the bound state of $(\mathbf{f},0)$ with $(\mathbf{f},\mathbf{f})$ in the large N limit.

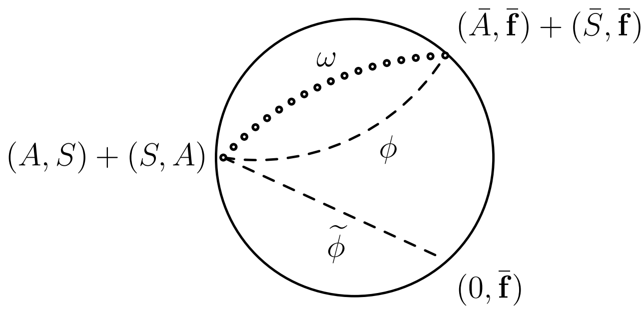
This is precisely consistent with the identifications

$$(0,\mathbf{f}) \sim \widetilde{\phi}, \qquad \frac{(A,\mathbf{f})+(S,\mathbf{f})}{\sqrt{2}} \sim \phi\omega, \qquad \frac{(A,S)+(S,A)}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}\left(\omega\phi\widetilde{\phi} - \frac{1}{\Delta_{(\mathbf{f},\mathbf{f})}}\partial\omega\bar{\partial}\omega\right). \tag{3.114}$$

The leading $\mathcal{O}(N^0)$ contribution to (3.113) comes from the free field contraction of

$$\left\langle \bar{\widetilde{\phi}} \ : \bar{\phi}\bar{\omega} : \ \frac{:\omega\phi\widetilde{\phi}:}{\sqrt{2}} \right\rangle. \tag{3.115}$$

This is shown in the following (bulk) picture



As the last example of this section, let us also observe the following three-point function:

$$C_{nor}\left((0,\bar{\mathbf{f}}), \frac{(\bar{A},\bar{\mathbf{f}})-(\bar{S},\bar{\mathbf{f}})}{\sqrt{2}}, \frac{(A,S)-(S,A)}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \mathcal{O}(\frac{1}{N}). \tag{3.116}$$

As argued earlier, the operator $\frac{1}{\sqrt{2}}((A,\mathbf{f})-(S,\mathbf{f}))$ is an elementary particle state; denote it by Ψ. We have $\Delta_\Psi = \Delta_{(\mathbf{f},0)}$ in the large N limit. Analogously, $\frac{1}{\sqrt{2}}((\mathbf{f},A)-(\mathbf{f},S)) = \widetilde{\Psi}$, with $\Delta_{\widetilde{\Psi}} = \Delta_{(0,\mathbf{f})}$ at large N. There is a similar three-point function, fusing $\phi = (\mathbf{f},0)$ and $\widetilde{\Psi}$ into $\frac{1}{\sqrt{2}}((S,A)-(A,S))$. Combining this with (3.116), we conclude that $\frac{1}{\sqrt{2}}((A,S)-(S,A))$ is a bound state of two elementary massive particles, namely

$$\frac{(A,S)-(S,A)}{\sqrt{2}} \sim \frac{\Psi\widetilde{\phi}-\widetilde{\Psi}\phi}{\sqrt{2}}. \tag{3.117}$$

3.5 Sphere four-point function

In the section, we investigate the sphere four-point function in the W_N minimal model, of the primary operators $(\mathbf{f}, 0)$, $(\bar{\mathbf{f}}, 0)$, with a general primary (Λ_+, Λ_-) and its charge conjugate. The main purpose of this exercise is to set things up for the torus two-point function in Section 3.6. We consider two different approaches in computing the sphere four-point function: the Coulomb gas formalism, and null state differential equations. In Section 3.5.1 through 3.5.3, we illustrate the screening charge contour integral and its relation with conformal blocks in various channels, primarily in the $N = 3$ example, i.e. the W_3 minimal model. In this case, the conformal blocks are computed by a two-fold contour integral on a sphere with four punches. More generally, the conformal blocks in the W_N minimal model are given by $(N-1)$-fold contour integrals. The identification of the correct contour for each conformal block, however, is not obvious for general N. In Section 3.5.4, we recall the null state differential equations of [43], which applies to all W_N minimal models. The conformal blocks are given by the N linearly independently solutions of the null state differential equation. One observes that the N distinct t-channel conformal blocks (to be defined below) are permuted under the action of the Weyl group. This motivates an identification of the Coulomb gas screening integral contours for the t-channel conformal blocks for all values of N, which we describe in Section 3.5.5. The monodromy invariance of our four-point functions is shown in Appendix 3.D.

3.5.1 Screening charges

Let us illustrate the screening charge integral in the W_3 minimal model. Consider the sphere four-point function of the primary operators $(\mathbf{f}, 0)$, $(\bar{\mathbf{f}}, 0)$, with a general primary

(Λ_+, Λ_-) and its charge conjugate. The highest weight vectors of $\mathbf{f}$ and $\bar{\mathbf{f}}$ are the two fundamental weights ω^1 and ω^2 of $SU(3)$. In the Coulomb gas approach, we first replace the four W_3 primaries by the corresponding chiral boson vertex operators $e^{iv_i \cdot X_L}$, $i = 1, 2, 3, 4$, where the charge vectors v_i are taken to be

$$v_1 = \sqrt{\frac{p'}{p}}\omega^1, \quad v_2 = \sqrt{\frac{p'}{p}}\omega^2, \quad v_3 = \sqrt{\frac{p'}{p}}\Lambda_+ - \sqrt{\frac{p}{p'}}\Lambda_-, \quad v_4 = 2Q - v_3. \tag{3.118}$$

There is some freedom in choosing the charge vectors, since different charge vectors related by the shifted Weyl transformations are identified with the same W-algebra primary. For instance, here we have chosen v_4 to be $2Q - v_3$ rather than $\bar{v}_3 = \sqrt{\frac{p'}{p}}\bar{\Lambda}_+ - \sqrt{\frac{p}{p'}}\bar{\Lambda}_-$. Indeed these two ways to represent the primary $(\bar{\Lambda}_+, \bar{\Lambda}_-)$ are related by the longest Weyl reflection, as explained at the end of Section 3.3. In terms of Dynkin labels, we write

$$\omega^1 = (1,0), \quad \omega^2 = (0,1), \quad Q = -\frac{1}{\sqrt{pp'}}(1,1), \quad \Lambda_+ = (n_+, m_+), \quad \Lambda_- = (n_-, m_-), \tag{3.119}$$

where $n_\pm, m_\pm$ are nonnegative integers that obey $n_+ + m_+ \leq k = p-3$, $n_- + m_- \leq k+1 = p-2$. The two simple roots are $\alpha_1 = (2,-1)$, $\alpha_2 = (-1,2)$. The corresponding simple Weyl reflections s_1, s_2 act on the weight vector (n, m) by

$$s_1(n,m) = (-n, n+m), \quad s_2(n,m) = (n+m, -m). \tag{3.120}$$

To compute the sphere four point function of the W_N primaries, we must insert screening charges so that the total charge is $2Q$. In our example, a total screening charge $-v_1 - v_2 = -\sqrt{\frac{p'}{p}}(\alpha_1 + \alpha_2)$ is inserted. This is done by inserting two screening operators, V_1^- and V_2^-, both of which have conformal weight 1. So we expect

$$\langle \mathcal{O}_{v_1}(x_1)\mathcal{O}_{v_2}(x_2)\mathcal{O}_{v_3}(x_3)\mathcal{O}_{v_4}(x_4)\rangle = \int_C ds_1 ds_2 \langle V_{v_1}(x_1)V_{v_2}(x_2)V_{v_3}(x_3)V_{v_4}(x_4)V_1^-(s_1)V_2^-(s_2)\rangle, \tag{3.121}$$

for some appropriate choice of the contour C for the (s_1, s_2)-integral. In fact, by choosing the appropriate contour C, we can pick out the three independent conformal blocks in this case. One may allow the contours to start and end on one of the x_i's where the vertex operator is inserted, but we will demand the contours are closed on the four-punctured sphere.[12] This will allow for a straightforward generalization to the torus two-point function later.

Without loss of generality, we will choose $x_3 = 0$, $x_4 = \infty$, while keeping x_1, x_2 two general points on the complex plane. Write $V'_{v_4}(\infty) = \lim_{x_4\to\infty} x_4^{2h_{v_4}} V_{v_4}(x_4)$. The correlation function with screening operators is computed in the free boson theory (with linear dilaton) as

$$\begin{aligned}
&\langle V_{v_1}(x_1)V_{v_2}(x_2)V_{v_3}(0)V'_{v_4}(\infty)V_1^-(s_1)V_2^-(s_2)\rangle \\
&= x_{12}^{v_1\cdot v_2} s_{12}^{\frac{p'}{p}\alpha_1\cdot\alpha_2} \prod_{i=1}^{2} x_i^{v_i\cdot v_3} s_i^{-\sqrt{\frac{p'}{p}}v_3\cdot\alpha_i} \prod_{i,j=1}^{2} (x_i - s_j)^{-\sqrt{\frac{p'}{p}}v_i\cdot\alpha_j} \\
&= x_1^{\frac{p'}{p}(\frac{2}{3}n_+ + \frac{1}{3}m_+) - (\frac{2}{3}n_- + \frac{1}{3}m_-)} x_2^{\frac{p'}{p}(\frac{1}{3}n_+ + \frac{2}{3}m_+) - (\frac{1}{3}n_- + \frac{2}{3}m_-)} s_1^{-\frac{p'}{p}n_+ + n_-} s_2^{-\frac{p'}{p}m_+ + m_-} \\
&\quad\times x_{12}^{\frac{p'}{3p}} s_{12}^{-\frac{p'}{p}} (x_1 - s_1)^{-\frac{p'}{p}} (x_2 - s_2)^{-\frac{p'}{p}}.
\end{aligned} \tag{3.122}$$

Note that as a function in s_1, (3.122) has branch points at $s_1 = 0, \infty, x_1, s_2$. As a function in s_2, it has branch points at $s_2 = 0, \infty, x_2, s_1$. The property that there are 4, rather than 5, branch points in each s_i, will be important in the construction of the contour C.

3.5.2 Integration contours

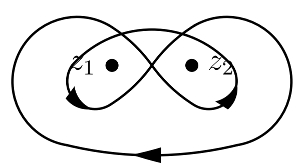
We will consider the following type of the two-dimensional integration contour C. First integrate s_2 along a contour $C_2(s_1)$ which depends on s_1, and then integrate s_1 along a contour C_1. $C_2(s_1)$ is chosen to avoid the four branch points $s_2 = 0, \infty, x_2, s_1$, and C_1 is

[12] Strictly speaking, due to the branch cuts connecting the vertex operators V_{v_i}, the contour C lies on a covering Riemann surface of the punctured sphere.

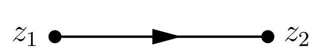
chosen to avoid the branch points $0, \infty, x_1, x_2$ (x_2 will be a branch point in s_1 after the integration over s_2). To ensure that one comes back to the same sheet by going once around the contour, we demand that C_1, C_2 have no net winding number around any branch point.[13]

For $C_2(s_1)$ to be well defined the entire time as s_1 moves along C_1, we also demand the following property of C_1: upon removal of the s_1-branch point x_1, C_1 becomes contractible. Since x_1 is not a branch point of the s_2-integrand, this makes it possible to choose $C_2(s_1)$ to avoid all branch points of s_2 and comes back to itself as s_1 goes around C_1, ensuring that the full contour integral is well defined.

Let us denote by $L(z_1, z_2)$ the following contour that goes around two points z_1, z_2 on the complex plane:



This contour is well defined when there are branch cuts coming out of z_1 and z_2, and the monodromies around z_1 and z_2 commute. It is also nontrivial only when z_1 and z_2 are both branch points. If we integrate (3.122) along a contour $L(z_1, z_2)$ where z_1, z_2 are two of the branch points of the integrand, the contour may be collapsed to a line interval connecting z_1 and z_2, namely



in the following sense. Let g_{z_1} and g_{z_2} be the action by the monodromy around z_1 and z_2

[13] This is sufficient because the monodromies involved are abelian.

respectively. Then we can write

$$\int_{L(z_1,z_2)} \cdots = (1 - g_{z_2} + g_{z_1} g_{z_2} - g_{z_2}^{-1} g_{z_1} g_{z_2}) \int_{z_1}^{z_2} \cdots \tag{3.123}$$

where an appropriate branch is chosen for the integral from z_1 to z_2 on the RHS.

The two-dimensional contour C will be constructed as follows: we first integrate s_2 along a contour $C_2(s_1)$ of the form $L(z_1, z_2)$, where z_1, z_2 are two out of the four branch points $0, \infty, x_2, s_1$, and then integrate s_1 along a contour C_1 that is of the form $L(x_1, z)$ (so that it becomes contractible upon removal of x_1). We must then investigate the transformation of the contour integral under the monodromies associated with s- and t-channel Dehn twists:

$$\begin{aligned} &T_s : \ x_1 \text{ going around } x_2, \text{ and} \\ &T_t : \ x_1 \text{ going around } 0. \end{aligned} \tag{3.124}$$

These are analyzed in detail in Appendix 3.B. We only describe the results below.

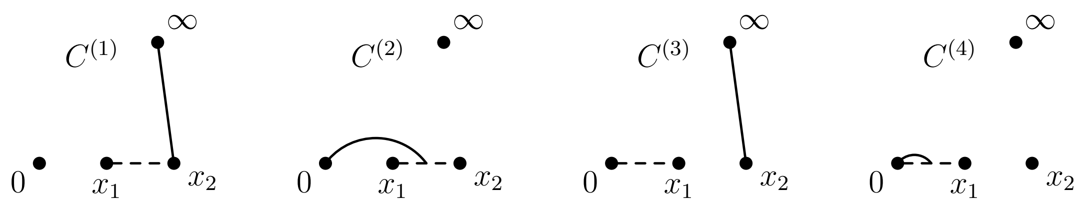
Among the following four L-contours for the s_2-integral: $L(x_2, \infty)$, $L(0, s_1)$, $L(0, \infty)$, and $L(x_2, s_1)$, only two are linearly independent. In fact, the basis $(L(x_2, \infty), L(0, s_1))$ is convenient for analyzing t-channel monodromies, whereas the basis $(L(0, \infty), L(x_2, s_1))$ is convenient for analyzing s-channel monodromies. The linear transformation between the two basis is given by

$$\begin{pmatrix} L(0,\infty) \\ L(x_2,\infty) \end{pmatrix} = \begin{pmatrix} \frac{1-g_{s_1} g_{x_2}}{1-g_{s_1}} & \frac{1-g_0}{1-g_{s_1}} \\ -g_{s_1} \frac{1-g_{x_2}}{1-g_{s_1}} & -\frac{1-g_0 g_{s_1}}{1-g_{s_1}} \end{pmatrix} \begin{pmatrix} L(0,s_1) \\ L(s_1,x_2) \end{pmatrix}. \tag{3.125}$$

Using the basis for the s_2-integral adapted to the t-channel, namely $(L(x_2, \infty), L(0, s_1))$, we may consider the following four candidates for the two-dimensional contour C,

$$\begin{aligned} &\int_{C^{(1)}} = \int_{L(x_1,x_2)} ds_1 \int_{L(x_2,\infty)} ds_2, \qquad \int_{C^{(2)}} = \int_{L(x_1,x_2)} ds_1 \int_{L(0,s_1)} ds_2, \\ &\int_{C^{(3)}} = \int_{L(0,x_1)} ds_1 \int_{L(x_2,\infty)} ds_2, \qquad \int_{C^{(4)}} = \int_{L(0,x_1)} ds_1 \int_{L(0,s_1)} ds_2. \end{aligned} \tag{3.126}$$

These contours are shown in the figures below:



The solid lines represents the interval onto which the s_2-contour collapses (as opposed to the contour itself), whereas the dashed lines represent the corresponding collapsing interval of the s_1-contour.

We will denote the integral of (3.122) along $C^{(i)}$ by $\mathcal{J}_i$, $i = 1, 2, 3, 4$. The t-channel monodromy T_t then acts on the basis vector $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4)$ by the matrix

$$M_t = g_0(x_1) \begin{pmatrix} \mathbf{1} & \mathbf{1} - g_{x_2}(s_1) \\ 0 & g_0(s_1) g_{x_1}(s_1) \end{pmatrix}, \tag{3.127}$$

while the s-channel monodromy T_s acts by the matrix

$$M_s = g_{x_2}(x_1) \begin{pmatrix} g_{x_1}(s_1) g_{x_2}(s_1) & 0 \\ g_{x_1}(s_1) - g_{x_1}(s_1) g_0(s_1) & \mathbf{1} \end{pmatrix}. \tag{3.128}$$

In both (3.127) and (3.128), $g_x(z)$ denotes the 2×2 monodromy matrix that acts on the s_1-integrand (after having done the s_2-integral) by taking the point z around x. The explicit form of $g_0(x_1)$, $g_{x_2}(x_1)$, $g_0(s_1)$, $g_{x_1}(s_1)$, $g_{x_2}(s_1)$ are given in Appendix 3.B.

3.5.3 The conformal blocks for $N = 3$

While we have constructed four candidates for the two-dimensional contour C (out of many possibilities), there are only three linearly independent conformal blocks for the four-point function considered in Section 3.5.1. Indeed, only three out of the four $\mathcal{J}_i$'s are linearly independent, as shown in Appendix 3.C. They are

$$\begin{pmatrix} \widetilde{\mathcal{J}}_2 \\ \mathcal{J}_3 \\ \mathcal{J}_4 \end{pmatrix} = \begin{pmatrix} \int_{L(x_1,x_2)} ds_1 \int_{L(s_1,x_2)} ds_2 \cdots \\ \int_{L(0,x_1)} ds_1 \int_{L(x_2,\infty)} ds_2 \cdots \\ \int_{L(0,x_1)} ds_1 \int_{L(0,s_1)} ds_2 \cdots \end{pmatrix}, \tag{3.129}$$

where the integrand $\cdots$ is given by (3.122).

There are three s-channel conformal blocks, corresponding to fusing the $(\mathbf{f}, 0)$ and $(\bar{\mathbf{f}}, 0)$ into $(0,0)$, $(adj, 0)$, and $(adj', 0)$, where adj stands for the adjoint representation of $SU(3)$, and adj' refers to a second adjoint W_3-conformal block whose lowest weight channel is the $(W^3)_{-1}$ descendant of $(adj, 0)$. We denote these conformal blocks by

$$\mathcal{F}^s = (\mathcal{F}^s(0), \mathcal{F}^s(adj), \mathcal{F}^s(adj')) \,. \tag{3.130}$$

The lowest conformal weights in these channels are (computed using (3.9))

$$\begin{aligned} h_{(\mathbf{f},0)} &= h_{(\bar{\mathbf{f}},0)} = \frac{N-1}{2N}(1+\frac{N+1}{N+k}) = \frac{4p'}{3p} - 1, \\ h_{(adj,0)} &= 1 + \frac{N}{N+k} = \frac{3p'}{p} - 2, \quad h_{(adj',0)} = \frac{3p'}{p} - 1. \end{aligned} \tag{3.131}$$

By comparing the s-channel monodromies, one finds that $\mathcal{F}^s$ is expressed in terms of the contour integrals via the linear transformation

$$\mathcal{F}^s = \mathcal{A}_s \begin{pmatrix} \widetilde{\mathcal{J}}_2 \\ \mathcal{J}_3 \\ \mathcal{J}_4 \end{pmatrix}, \quad \mathcal{A}_s = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\zeta^{2-m_+-n_+}(1-\zeta^{2+m_+})(1-\zeta^{n_+})}{(1-\zeta)^2(1+\zeta)(1+\zeta+\zeta^2)} & 1 & 0 \\ -\frac{\zeta^{3-2m_+-n_+}(1-\zeta^{m_+})(1-\zeta^{1+m_++n_+})}{(1-\zeta)^2(1+\zeta)(1+\zeta+\zeta^2)} & 0 & 1 \end{pmatrix}, \tag{3.132}$$

where $\zeta \equiv e^{2\pi i p'/p}$.

Similarly, in the t-channel, there are three conformal blocks, associated with three distinct primaries $(\Lambda_+ + \omega^1, \Lambda_-)$, $(\Lambda_+ - \omega^1 + \omega^2, \Lambda_-)$, and $(\Lambda_+ - \omega^2, \Lambda_-)$. The conformal blocks are denoted

$$\mathcal{F}^t = \left(\mathcal{F}^t(\omega^1), \mathcal{F}^t(-\omega^1+\omega^2), \mathcal{F}^t(-\omega^2)\right). \tag{3.133}$$

The lowest conformal weights in the respective channels are

$$\begin{aligned}
h_{(\Lambda_+ + \omega^1, \Lambda_-)} &= h_{(\Lambda_+, \Lambda_-)} + \frac{p'}{p}(\frac{2}{3}n_+ + \frac{1}{3}m_+ + \frac{4}{3}) - \frac{2}{3}n_- - \frac{1}{3}m_- - 1, \\
h_{(\Lambda_+ - \omega^1 + \omega^2, \Lambda_-)} &= h_{(\Lambda_+, \Lambda_-)} + \frac{p'}{p}(-\frac{1}{3}n_+ + \frac{1}{3}m_+ + \frac{1}{3}) + \frac{1}{3}n_- - \frac{1}{3}m_-, \\
h_{(\Lambda_+ - \omega^2, \Lambda_-)} &= h_{(\Lambda_+, \Lambda_-)} + \frac{p'}{p}(-\frac{1}{3}n_+ - \frac{2}{3}m_+ - \frac{2}{3}) + \frac{1}{3}n_- + \frac{2}{3}m_- + 1.
\end{aligned} \tag{3.134}$$

By comparing with the t-channel monodromy, we find that $\mathcal{F}^t$ is expressed in terms of the contour integrals as

$$\mathcal{F}^t = \mathcal{A}_t \begin{pmatrix} \widetilde{\mathcal{J}}_2 \\ \mathcal{J}_3 \\ \mathcal{J}_4 \end{pmatrix}, \quad \mathcal{A}_t = \begin{pmatrix} 1 & -\frac{(1-\zeta)^2(1+\zeta)\zeta^{-1+m_++n_+}}{(1-\zeta^{1+m_+})(1-\zeta^{1+n_+})} & -\frac{(1-\zeta)^2(1+\zeta)\zeta^{-1+2m_++n_+}}{(1-\zeta^{1+m_+})(1-\zeta^{2+m_++n_+})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.135}$$

Finally, the four point function is obtained by summing over either the s-channel or the t-channel conformal blocks,

$$\langle \mathcal{O}_{v_1}(x_1,\bar{x}_1)\mathcal{O}_{v_2}(x_2,\bar{x}_2)\mathcal{O}_{v_3}(0)\mathcal{O}'_{v_4}(\infty)\rangle = (\mathcal{F}^s)^\dagger \mathcal{M}^s \mathcal{F}^s = (\mathcal{F}^t)^\dagger \mathcal{M}^t \mathcal{F}^t. \tag{3.136}$$

Here $\mathcal{M}^s$ and $\mathcal{M}^t$ are "mass" matrices, and obey

$$(\mathcal{A}^s)^\dagger \mathcal{M}^s \mathcal{A}^s = (\mathcal{A}^t)^\dagger \mathcal{M}^t \mathcal{A}^t. \tag{3.137}$$

$\mathcal{M}^t$ is diagonal, while $\mathcal{M}^s$ is only block diagonal a priori, since there are two adjoint conformal blocks in the s-channel. The mass matrices are computed explicitly in Appendix 3.C, up to the overall normalization which can be fixed by the identity s-channel. In this way, the four point function is entirely determined.

3.5.4 Null state differential equations

In this section, we describe a different method of computing the sphere four-point function of the W_N primaries $(\mathbf{f}, 0)$, $(\bar{\mathbf{f}}, 0)$ with (Λ_+, Λ_-) and its charge conjugate, following [43]. Analogously to Section 3.5.1, now for general N, the four operator on the sphere are $\mathcal{O}_{v_i}$ with the charge vectors v_i given by

$$v_1 = \sqrt{\frac{p'}{p}}\omega^1, \quad v_2 = \sqrt{\frac{p'}{p}}\omega^{N-1}, \quad v_3 = v \equiv \sqrt{\frac{p'}{p}}\Lambda_+ - \sqrt{\frac{p}{p'}}\Lambda_-, \quad v_4 = 2Q - v. \tag{3.138}$$

To compare with the formulae in Section 3.3, we also write

$$u = \lambda + \lambda' = v - Q, \tag{3.139}$$

where λ and λ' lie in the lattices $\Gamma^*_{p/p'}$ and $\Gamma^*_{p'/p}$, and are defined modulo simultaneous shifts by lattice vectors of $\Gamma_{pp'}$ with the opposite signs. As shown in [43], the primary states $(\mathbf{f}, 0)$ and $(\bar{\mathbf{f}}, 0)$ are complete degenerate. They obey a set of null state equations. For instance, in the W_3 minimal model, the vertex operators $\mathcal{O}_{v_1}$ gives rise to the null states

$$\begin{aligned}
&\left(W_{-1} - \frac{3w}{2\Delta}L_{-1}\right)\mathcal{O}_{v_1} = 0,\\
&\left(W_{-2} - \frac{12w}{\Delta(5\Delta+1)}L_{-1}^2 + \frac{6w(\Delta+1)}{\Delta(5\Delta+1)}L_{-2}\right)\mathcal{O}_{v_1} = 0,\\
&\left(W_{-3} - \frac{16w}{\Delta(\Delta-1)(5\Delta+1)}L_{-1}^3 + \frac{12w}{\Delta(5\Delta+1)}L_{-1}L_{-2} + \frac{3w(\Delta-3)}{2\Delta(5\Delta+1)}L_{-3}\right)\mathcal{O}_{v_1} = 0.
\end{aligned} \tag{3.140}$$

Here Δ and w are the conformal weight and spin-3 charge of $\mathcal{O}_{v_1}$. Explicitly, they are given by

$$\Delta = \frac{4p'}{3p} - 1, \qquad w^2 = -\frac{2\Delta^2}{27}\frac{5p'-3p}{3p-5p}. \tag{3.141}$$

Similar relations hold for $\mathcal{O}_{v_2}$. Using the null state equations, one finds that in the W_3 minimal model the conformal blocks obey hypergeometric differential equation of $(3, 2)$-type.

The null state method applies straightforwardly to the W_N minimal model with general N, and the conformal blocks therein obey the following hypergeometric differential equation of $(N, N-1)$-type:

$$\left[x\prod_{k=1}^{N}\left(x\frac{d}{dx}+\frac{p'}{p}+\sqrt{\frac{p'}{p}}P_{1,k}\right)-\prod_{k=1}^{N}\left(x\frac{d}{dx}+\sqrt{\frac{p'}{p}}P_{1,k}\right)\right]\mathcal{G}(x)=0, \tag{3.142}$$

where x is the conformally invariant cross ratio of the four x_i's, and $P_{i,j}$ are defined in terms of the charge vectors as

$$P_k = u\cdot \mathbf{h}_k, \qquad P_{ij} = P_i - P_j. \tag{3.143}$$

The vectors $\mathbf{h}_k$ were defined in (3.77). The solutions to (3.142) are

$$\mathcal{G}_k(x) = x^{\sqrt{\frac{p'}{p}}P_{k,1}}\,{}_NF_{N-1}(\vec{\mu}_k;\widehat{\vec{\nu}}_k|x) \equiv x^{-\sqrt{\frac{p'}{p}}P_1}G_k(x). \tag{3.144}$$

where $\vec{\mu}_k$ and $\vec{\nu}_k$ are the following N-dimensional vectors:

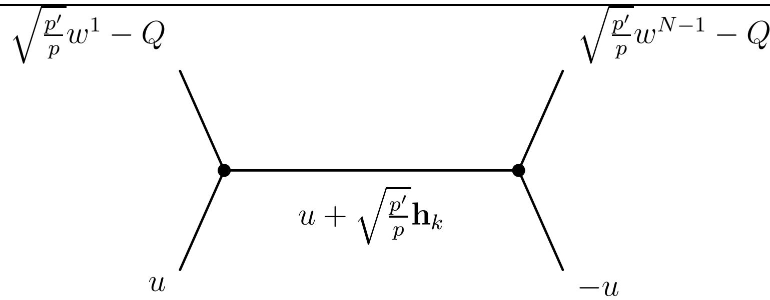
$$\begin{aligned}\vec{\mu}_k &= \sqrt{\frac{p'}{p}}(P_{k,1},\cdots,P_{k,N})+\frac{p'}{p}(1,\cdots,1),\\ \vec{\nu}_k &= \sqrt{\frac{p'}{p}}(P_{k,1},\cdots,P_{k,N})+(1,\cdots,1),\end{aligned} \tag{3.145}$$

and $\widehat{\vec{\nu}}_k$ is the $(N-1)$-dimensional vector defined by dropping the k-th entry of $\vec{\nu}_k$. ${}_NF_{N-1}(a_1,\cdots,a_N;b_1,\cdots,$ is the generalized hypergeometric function.

One observes that, the action of shifted Weyl transformations on v (or equivalently, ordinary Weyl transformation on u) permutes the N t-channel conformal blocks. One may define a Weyl group action on P_k as

$$w(P_k) = w(u)\cdot\mathbf{h}_k = u\cdot w^{-1}(\mathbf{h}_k). \tag{3.146}$$

The Weyl group acts as permutations on $\mathbf{h}_k$, and hence permutes P_k and $G_k(x)$ as well. Diagrammatically, the t-channel conformal blocks can be represented as



The shifted Weyl transformation on v permutes the diagrams with different internal lines.

In terms of the conformal blocks $\mathcal{G}_k(x)$ or $G_k(x)$, the four-point function is given by

$$\begin{aligned}&\left\langle \mathcal{O}_{v_1}(x_1)\mathcal{O}_{v_2}(x_2)\mathcal{O}_{v_3}(0)\mathcal{O}'_{v_4}(\infty)\right\rangle \\ &= |x_1 - x_2|^{\frac{2p'}{Np}} |x_1|^{2\sqrt{\frac{p'}{p}}Q\cdot\mathbf{h}_1} |x_2|^{-2\sqrt{\frac{p'}{p}}Q\cdot\mathbf{h}_N - 2\frac{p'}{p}} G\left(\frac{x_1}{x_2}, \frac{\bar{x}_1}{\bar{x}_2}\right).\end{aligned} \tag{3.147}$$

where $G(x,\bar{x})$ sums up the product of holomorphic and anti-holomorphic conformal blocks,

$$G(x,\bar{x}) = \sum_{j=1}^{N} (\mathcal{M}_u)_{jj} G_j(x) G_j(\bar{x}). \tag{3.148}$$

$\mathcal{M}_u$ is a diagonal "mass matrix". We indicated here the explicit u-dependence of $\mathcal{M}_u$, though $G_j(x)$ depend on u as well. $\mathcal{M}_u$ can be expressed in terms of the structure constants (three point function coefficients) via

$$\begin{aligned}(\mathcal{M}_u)_{jj} &= B\left(\sqrt{\frac{p'}{p}}w^1\right)^2 C_{W_N}\left(\sqrt{\frac{p'}{p}}w^1, u+Q, Q-u-\sqrt{\frac{p'}{p}}\mathbf{h}_j\right) \\ &\quad \times C_{W_N}\left(Q+u+\sqrt{\frac{p'}{p}}\mathbf{h}_j, \sqrt{\frac{p'}{p}}w^{N-1}, Q-u\right) \\ &= \gamma\left(\frac{p'}{p}\right)\gamma\left(N\left(1-\frac{p'}{p}\right)\right)\prod_{i=1, i\neq j}^{N}\gamma\left(\sqrt{\frac{p'}{p}}P_{ij}\right)\gamma\left(\frac{p'}{p}-\sqrt{\frac{p'}{p}}P_{ij}\right).\end{aligned} \tag{3.149}$$

In deriving the last line, we used the results of B and C_{W_N} computed in Section 3.4. Note that, expectedly, the Weyl transformations on u also permutes the N diagonal entries of $\mathcal{M}_u$. For later use, we also define

$$C_u^2 \equiv (\mathcal{M}_u)_{N,N} = \gamma\left(\frac{p'}{p}\right)\gamma\left(N\left(1-\frac{p'}{p}\right)\right)\prod_{i=1}^{N-1}\gamma\left(\sqrt{\frac{p'}{p}}P_{i,N}\right)\gamma\left(\frac{p'}{p}-\sqrt{\frac{p'}{p}}P_{i,N}\right). \tag{3.150}$$

3.5.5 The contour for general N

Let us return to the Coulomb gas formalism, and we are now ready to present a contour prescription for the four-point conformal blocks in W_N minimal models with general N. It may appear rather difficult to directly identify the N contours that give precisely the N linearly independent conformal blocks. But once we find the contour that gives one of the N t-channel conformal blocks, we can apply Weyl transformations on the charge vector u and generate the remaining $N-1$ t-channel conformal blocks.

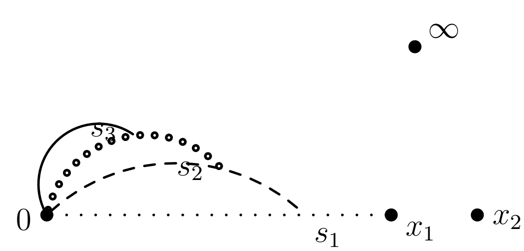
The screening charge integral that computes the four point function, or rather, a conformal block, takes the form

$$\begin{aligned}\mathbf{G}_u\left(\frac{x_1}{x_2}\right) =& x_2^{\frac{p'}{p}-\sqrt{\frac{p'}{p}}P_N} x_1^{\sqrt{\frac{p'}{p}}P_1} \oint ds_1\, s_1^{-\sqrt{\frac{p'}{p}}(u+Q)\cdot\alpha_1} (x_1-s_1)^{-\frac{p'}{p}} \\ &\times\left(\prod_{i=1}^{N-2}\oint ds_{i+1}\, s_{i+1}^{-\sqrt{\frac{p'}{p}}(u+Q)\cdot\alpha_{i+1}} (s_i-s_{i+1})^{-\frac{p'}{p}}\right)(x_2-s_{N-1})^{-\frac{p'}{p}}\end{aligned} \tag{3.151}$$

where $s_1, s_2, \cdots, s_{N-1}$ are integrated along the following choice of contour:

$$\prod_{i=1}^{N-1}\oint ds_i = \int_{L(0,x_1)} ds_1 \int_{L(0,s_1)} ds_2 \cdots \int_{L(0,s_{N-2})} ds_{N-1}. \tag{3.152}$$

Pictorially, this is represented as



where the various lines represent the collapsing intervals of the L-contours of $s_1, s_2, s_3, \cdots$. In the $N=3$ case, this is the last contour of (3.126), denoted by $C^{(4)}$ in Section 3.5.2.

The integral (3.151) can be computed by collapsing the prescribed contour to successive integrations over straight lines,

$$\int_{L(0,x_1)} ds_1 \int_{L(0,s_1)} ds_2 \cdots \int_{L(0,s_{N-2})} ds_{N-1} = \mathcal{N}_u \int_0^{x_1} ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{N-2}} ds_{N-1}, \tag{3.153}$$

where the factor $\mathcal{N}_u$ is obtained by taking the differences of line integrals related by monodromies, similarly to the derivation in Appendix 3.B. The result is

$$\begin{aligned}
\mathcal{N}_u &= \prod_{i=1}^{N-1} (1 - g_{s_i})(1 - g_{0,i}), \\
g_{s_i} &= e^{-2\pi i \frac{p'}{p}}, \\
g_{0,N-i} &= e^{-2\pi i \sqrt{\frac{p'}{p}} u \cdot \sum_{j=1}^{i-1} \alpha_{N-j} - 2\pi i \sqrt{\frac{p'}{p}} (u+Q)\cdot \alpha_{N-i}} = e^{2\pi i (-\sqrt{\frac{p'}{p}} P_{i,N} + \frac{p'}{p})}.
\end{aligned} \tag{3.154}$$

The integral expression $\mathbf{G}_u$ is related to the conformal block $G_N(x)$ derived in the previous subsection as

$$\begin{aligned}
\mathbf{G}_u\left(\frac{x_1}{x_2}\right) =& \mathcal{N}_u x_2^{\frac{p'}{p} - \sqrt{\frac{p'}{p}} P_N} x_1^{\sqrt{\frac{p'}{p}} P_1} \int_0^{x_1} ds_1 \; s_1^{-\sqrt{\frac{p'}{p}}(u+Q)\cdot\alpha_1} (x_1 - s_1)^{-\frac{p'}{p}} \\
& \times \left(\prod_{i=1}^{N-2} \int_0^{s_i} ds_{i+1} \; s_{i+1}^{-\sqrt{\frac{p'}{p}}(u+Q)\cdot\alpha_{i+1}} (s_i - s_{i+1})^{-\frac{p'}{p}} \right) (x_2 - s_{N-1})^{-\frac{p'}{p}} \\
=& \mathcal{N}_u \frac{\prod_{k=1}^{N-1} \Gamma(\sqrt{\frac{p'}{p}} P_{N,k} + \frac{p'}{p})}{\prod_{k=1}^{N-1} \Gamma(\sqrt{\frac{p'}{p}} P_{N,k} + 1)} \Gamma(1 - \frac{p'}{p})^{N-1} G_N\left(\frac{x_1}{x_2}\right) \\
\equiv& \mathcal{N}_u L_u G_N\left(\frac{x_1}{x_2}\right),
\end{aligned} \tag{3.155}$$

i.e. they differ only by the normalization constant $\mathcal{N}_u L_u$. Here we made use of the integral representation of the generalized hypergeometric function:

$$\begin{aligned}
&{}_N F_{N-1}(a_1, \cdots, a_N; b_1, \cdots, b_{N-1} | x) \\
&= \left(\prod_{k=1}^{N-1} \frac{\Gamma(b_k)}{\Gamma(a_k)\Gamma(b_k - a_k)} \right) \int_0^1 \cdots \int_0^1 \prod_{k=1}^{N-1} \xi_k^{a_k - 1} (1 - \xi_k)^{b_k - a_k - 1} \left(1 - x \prod_{k=1}^{N-1} \xi_k \right)^{-a_N} d\xi_1 \cdots d\xi_{N-1}.
\end{aligned} \tag{3.156}$$

Now we have obtained the N-th t-channel conformal block of Section 3.5.4. To produce the other t-channel conformal blocks, we act the Weyl transformation on u, and obtain

$$G_i\left(\frac{x_1}{x_2}\right) = G_N\left(\frac{x_1}{x_2}\right)\Big|_{u\to w(u)} = \mathcal{N}_{w(u)}^{-1} L_{w(u)}^{-1} \mathbf{G}_{w(u)}\left(\frac{x_1}{x_2}\right). \tag{3.157}$$

In terms of the contour integral $\mathbf{G}_u(x)$, the four-point function (3.148) can be written as

$$G(x,\bar{x}) = \frac{1}{(N-1)!}\sum_{w\in W} |\mathcal{C}_{w(u)}\mathbf{G}_{w(u)}(x)|^2, \tag{3.158}$$

where we defined the normalization constant $\mathcal{C}_u$ as

$$\mathcal{C}_u = C_u L_u^{-1}\mathcal{N}_u^{-1}. \tag{3.159}$$

A useful formula, derived using (3.150), is

$$C_u^2 L_u^{-2} = -\,\Gamma(1-\frac{p'}{p})^{2-2N}\gamma\left(\frac{p'}{p}\right)\gamma\left(N\left(1-\frac{p'}{p}\right)\right)\prod_{k=1}^{N-1}\csc\pi\sqrt{\frac{p'}{p}}P_{k,N}\sin\pi\left(\sqrt{\frac{p'}{p}}P_{k,N}-\frac{p'}{p}\right). \tag{3.160}$$

The representation of the four-point function (3.158) is the main result of this section. It may seen rather unnecessary given that we already know the relatively simple expression for the conformal blocks as generalized hypergeometric functions. But as discussed in the next section, our t-channel contour prescription allows for a straightforward generalization to torus two-point functions.

3.6 Torus two-point function

3.6.1 Screening integral representation

We now consider the torus two-point function of a fundamental primary and an anti-fundamental primary operator in the W_N minimal model, $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_2}$. The relevant genus one conformal blocks will be constructed using free bosons on the Narain lattice $\Gamma^{N-1,N-1}$, with insertions of vertex operators V_{v_1} and V_{v_2}, along with screening operators $V_1^-, V_2^-, \ldots, V_{N-1}^-$. Note that the set of screening operators is the same as in the earlier computation of sphere four point function, now the total charge being 0 on the torus (as opposed to $2Q$ on the sphere).

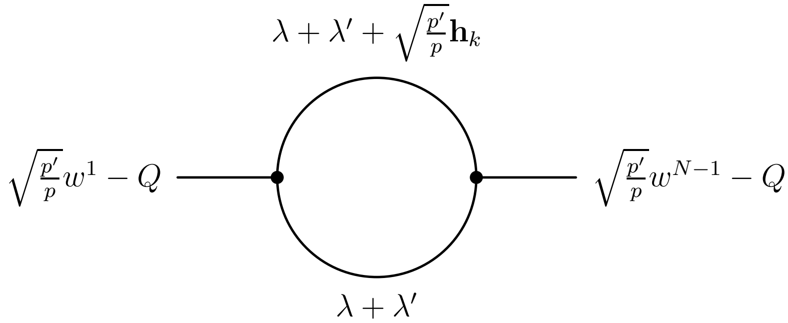
Our starting point is the torus correlation function in the free boson theory with screening operators insertions,

$$\begin{aligned}
&Z^{bos}_{\Gamma^{N-1,N-1}}\langle V_{v_1}(z_1)V_{v_2}(z_2)V_1^-(t_1)\cdots V_{N-1}^-(t_{N-1})\rangle_\tau \\
&= \frac{1}{|\eta(\tau)|^{2N-2}}\left|\frac{\theta_1(z_{12}|\tau)}{\partial_z\theta_1(0|\tau)}\right|^{2v_1\cdot v_2}\left|\frac{\theta_1(z_1-t_1|\tau)}{\partial_z\theta_1(0|\tau)}\right|^{-2\frac{p'}{p}}\left|\frac{\theta_1(z_2-t_{N-1}|\tau)}{\partial_z\theta_1(0|\tau)}\right|^{-2\frac{p'}{p}}\prod_{i=1}^{N-2}\left|\frac{\theta_1(t_{i,i+1}|\tau)}{\partial_z\theta_1(0|\tau)}\right|^{2\frac{p'}{p}\alpha_i\cdot\alpha_{i+1}} \\
&\quad\times\sum_{(v,\bar v)\in\Gamma^{N-1,N-1}} q^{\frac{1}{2}v^2}\bar q^{\frac{1}{2}\bar v^2}\exp\left[2\pi i\left(v\cdot(v_1z_1+v_2z_2-\sqrt{\frac{p'}{p}}\sum_{i=1}^{N-1}\alpha_i t_i)\right.\right. \\
&\qquad\qquad\qquad\qquad\left.\left.-\bar v\cdot(v_1\bar z_1+v_2\bar z_2-\sqrt{\frac{p'}{p}}\sum_{i=1}^{N-1}\alpha_i\bar t_i)\right)\right] \\
&= \sum_{u\in\Gamma^*_{pp'}/\Gamma_{pp'}}\left|G^{bos}_u(z_1,z_2,t_1,\cdots,t_{N-1}|\tau)\right|^2.
\end{aligned} \tag{3.161}$$

Our convention is that the coordinate z on the torus of modulus τ is identified under $z\sim z+1\sim z+\tau$. The lattice $\Gamma^{N-1,N-1}$ is defined as in (3.24). G^{bos}_u is a genus one character of the free boson with $N+1$ vertex operator insertions,

$$\begin{aligned}
&G^{bos}_u(z_1,z_2,t_1,\cdots,t_{N-1}|\tau) \\
&= \frac{1}{\eta(\tau)^{N-1}}\left(\frac{\theta_1(z_{12}|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{\frac{p'}{pN}}\left(\frac{\theta_1(z_1-t_1|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{-\frac{p'}{p}}\left(\frac{\theta_1(z_2-t_{N-1}|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{-\frac{p'}{p}}\prod_{i=1}^{N-2}\left(\frac{\theta_1(t_{i,i+1}|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{-\frac{p'}{p}} \\
&\quad\times\sum_{n\in\Gamma_{pp'}} q^{\frac{1}{2}(u+n)^2}\exp\left[2\pi i\sqrt{\frac{p'}{p}}\left((u+n)\cdot(\omega_1z_1+\omega_{N-1}z_2-\alpha_i t_i)\right)\right].
\end{aligned} \tag{3.162}$$

Recall that in the formula for the W_N minimal character (3.28), an alternating sum over Weyl orbits is perfomed in order to cancel the contribution from null states in the conformal family of $u = \lambda + \lambda'$ at the level $h_{w(\lambda)+\lambda'} - h_{\lambda+\lambda'}$ and higher. A similar procedure is applied here to produce the correct minimal W_N torus correlation function. A t-channel conformal block for the torus two-point function can be represented by the following diagram:



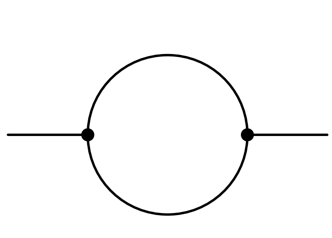
On the lower arc, there are null states at the level $h_{\lambda+w(\lambda')} - h_{\lambda+\lambda'}$ that are included by the free boson character. On the upper arc, there are null states at the level[14] $h_{\lambda+w(\lambda')+\sqrt{\frac{p'}{p}}\mathbf{h}_k} - h_{\lambda+\lambda'+\sqrt{\frac{p'}{p}}\mathbf{h}_k}$. To cancel the contribution from these null states, we consider the alternating sum:[15]

$$\mathcal{G}^{bos}_{\lambda+\lambda'}(z_1, z_2, t_1, \cdots, t_{N-1}|\tau) = \sum_{w\in W} \epsilon(w) G^{bos}_{\lambda+w(\lambda')}(z_1, z_2, t_1, \cdots, t_{N-1}|\tau). \tag{3.163}$$

Next, we integrate the positions t_i of the screening operators on an $(N-1)$-dimensional contour. Different appropriate contour choices may give different conformal blocks, say in the t-channel or s-channel.

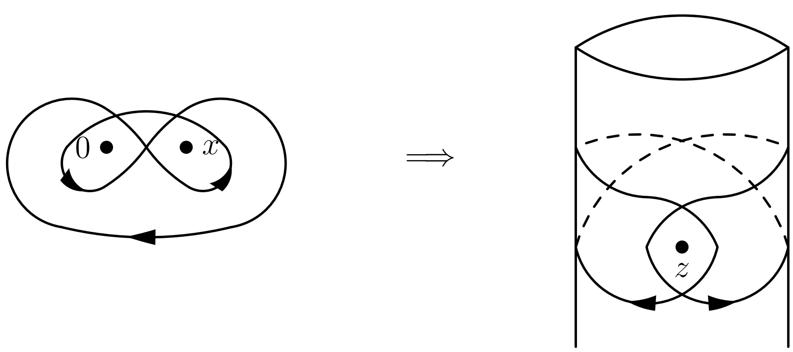
[14]Similar to (3.42), one can show that $h_{\lambda+w(\lambda')+\sqrt{\frac{p'}{p}}\mathbf{h}_k} - h_{\lambda+\lambda'+\sqrt{\frac{p'}{p}}\mathbf{h}_k}$ is always a nonnegative integer, when $\lambda + \sqrt{\frac{p'}{p}}\mathbf{h}_k$ and λ' sit in the identity affine Weyl chamber of $\Gamma^*_{\frac{p}{p'}}$ and $\Gamma^*_{\frac{p'}{p}}$.

[15]The reason that we are summing over the Weyl orbits of λ' (rather than, say λ) has to do with the inserted vertex operator being $(\mathbf{f}, 0)$ rather than $(0, \mathbf{f})$. Also note that normalization factors involving the structure constants, e.g. (3.160) are needed to obtain the full correlator. In fact, (3.160) is invariant under the Weyl transformation acting on λ', i.e. $C_{\lambda+w(\lambda')}L^{-1}_{\lambda+w(\lambda')} = C_{\lambda+\lambda'}L^{-1}_{\lambda+\lambda'}$. This is consistent with the W_N primaries being labelled by $u = \lambda + \lambda'$ up to the double Weyl action.



t-channel $\qquad\qquad$ s-channel

As in the case of sphere four-point function, we will construct the integration contour by composing one-dimensional contours with no net winding numbers, which ensures that the integral is well defined despite the branch cuts in the integrand. To go from the four-punctured sphere to the two-punctured torus, we can simply cut out holes around the points 0 and ∞ on the complex plane, and glue the two boundaries of resulting annulus to form the torus. The annulus coordinate x to the torus coordinate z are related by the exponential map $x = e^{2\pi i z}$. The L-contours introduced in Section 3.5.2 are closed contours that avoids the branch cuts including 0 and ∞, and thus are readily extended to the case of the torus under the exponential map. In particular, the part of the contour that winds around 0 or ∞ now winds around cycles of the torus.



We will still use $L(0,x)$ or $L(\infty,x)$ to denote the contour on the torus related by the exponential map, with the understanding that when the L-contour winds around 0 or ∞ on the plane, it now winds around the spatial cycle either above or below $z = \frac{1}{2\pi i}\log x$ on the torus.

Let us consider the following contour integral:

$$\mathcal{G}_u^t(z_1, z_2|\tau) = \int_{L(0,z_1)} dt_1 \int_{L(0,t_1)} dt_2 \cdots \int_{L(0,t_{N-2})} dt_{N-1} \mathcal{G}_u^{bos}(z_1, z_2, t_1, \cdots, t_{N-1}|\tau), \qquad (3.164)$$

which, as in the case of sphere four-point function, is a conformal block in t-channel. The contours $L(0, z_1)$, $L(0, t_1)$, $\cdots$, $L(0, t_{N-2})$, for $t_1, \cdots, t_{N-1}$ integrals, are now contours on the torus of the type shown in the right figure above. The positions of the two primaries, z_1, z_2 and the positions of the screening charges t_i, are in cylinder coordinates. They are related to x_1, x_2 and s_i described in Section 3.5.5, now annulus coordinates, by the conformal map

$$x_i = e^{2\pi i z_i}, \qquad s_i = e^{2\pi i t_i}. \qquad (3.165)$$

Generally, it appears rather difficult to explicitly identify a set of contours that gives all the conformal blocks in one channel. Instead, we use the trick described in Section 3.5.5, starting from (3.164) and obtain the other $N-1$ t-channel contours by Weyl transformation on $u = \lambda + \lambda'$. Note that in arriving at (3.164) we have already performed an alternating sum on λ', so the Weyl transformations that permute the different t-channel conformal blocks really only act on λ.

The torus two-point function of the primaries $(\mathbf{f}, 0)$ and $(\bar{\mathbf{f}}, 0)$ is then given by

$$\langle \mathcal{O}_{v_1}(z_1, \bar{z}_1)\mathcal{O}_{v_2}(z_2, \bar{z}_2)\rangle_\tau = \frac{1}{N!} \sum_{\lambda\in\Delta_1,\ \lambda'\in\Delta_2,\ w\in W} \left|\mathcal{C}_{w(u)}\mathcal{G}^t_{w(\lambda+\lambda')}(z_1, z_2|\tau)\right|^2, \qquad (3.166)$$

where Δ_1 and Δ_2 are the identity chambers of the shifted affine Weyl transformation in the lattices $\Gamma^*_{\frac{p}{p'}}$ and $\Gamma^*_{\frac{p'}{p}}$ respectively. In summing λ and λ' independently, we have overcounted, as (λ, λ') are identified under (3.21). This is compensated by including an extra factor of

$1/N$, turning the factor $\frac{1}{(N-1)!}$ in (3.158) into $\frac{1}{N!}$ in (3.166). The normalization factor $\mathcal{C}_u$ was given in (3.159) and (3.160).

3.6.2 Monodromy and modular invariance

On the torus with two operators inserted at x_1 and x_2, besides the s-monodromy (x_1 circling around x_2), t-monodromy ($x_1 \to x_1 + 1$ below x_2), and u-monodromy ($x_1 \to x_1 + 1$ above x_2), there are also what we may call the "v-monodromy" which is $x_1 \to x_1 + \tau$ on the left of x_2, and "w-monodromy" which is $x_1 \to x_1 + \tau$ on the right of x_2. Three of these five monodromies are independent. The two-point function should be invariant under these three monodromy transformations, as well as the modular transformations ($T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$).

The t-channel conformal blocks in (3.166) are trivially invariant under the t-monodromy and T-modular transformation. The s- and u-monodromy, on the other hand, mix the different t-channel conformal blocks. The invariance of the full two-point function can be seen by expanding (3.166) in powers of $q = e^{2\pi i\tau}$ with $z_1 - z_2$ fixed, where each term in the expansion is a sphere four-point function of $\mathcal{O}_{v_1}, \mathcal{O}_{v_2}$ with a pair of conjugate W_N primaries, or their decedents. The s- and u-monodromy invariance then follow from those of the sphere four-point functions.

The S-modular invariance is less obvious in terms of the t-channel conformal blocks. On the other hand, it acts in a simple way on the s-channel conformal blocks, and in particular leaves the identity channel invariant. The identity s-channel conformal block for the torus two-point function can be constructed by an easy generalization of the $\widetilde{J}_2$ contour in the $N = 3$ case for the sphere four-point function.

3.6.3 Analytic continuation to Lorentzian signature

As a potential application of the exact torus two-point function, we wish to consider its analytic continuation (with $\tau = i\beta$) to Lorentzian signature. The result is the Lorentzian thermal two-point function $\langle \mathcal{O}_{v_1}(t)\mathcal{O}_{v_2}(0)\rangle_\beta$ of the W_N minimal model on the circle (in our convention of z-coordinate, of circumference 1), at temperature $T = 1/\beta$. This Lorentzian two-point function measures the response of the system some time after the initial perturbation (by one of the two operators), and its decay in time would indicate thermalization of the perturbed system. Of course, since all operator scaling dimensions in the W_N minimal model are multiples of $\frac{1}{Npp'} = \frac{1}{N(N+k)(N+k+1)}$, Poincaré recurrence must occur at time $t = Npp' \sim N^3$. In fact, we will see that it occurs at time $t = Np$ in the two-point function $\langle \mathcal{O}_{v_1}(t)\mathcal{O}_{v_2}(0)\rangle_\beta$. Nonetheless, the behavior of the two-point function at time of order N^0 in the large N limit should be a useful probe of the dual semi-classical bulk geometry.

For simplicity of notation, we will denote both $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_2}$ by $\mathcal{O}$ in most of the discussion below, thinking of $\mathcal{O}$ as a real operator. Starting with the Euclidean torus two-point function $\langle \mathcal{O}(z,\bar{z})\mathcal{O}(0,0)\rangle_\tau$, we can write

$$z = x + iy, \qquad \bar{z} = x - iy, \tag{3.167}$$

and then at least locally make the Wick rotation y to $-it$. In other words, we would like to make the replacement

$$z \to x + t, \qquad \bar{z} \to x - t. \tag{3.168}$$

The resulting two-point function has a singularity at $x = \pm t$, when the two operators are light-like separated (as null rays go around the cylinder periodically, the two operators are light-like separated also when $x \pm t$ is an integer[16]). One must then specify how one wishes to analytically continue from $t < |x|$ to $t > |x|$. If we are interested in the time-ordered two-point function at $t > 0$,

$$\begin{aligned}\langle \mathcal{T}\mathcal{O}(x,t)\mathcal{O}(0)\rangle_\beta &= \sum_n e^{-\beta E_n}\langle n|\mathcal{T}\mathcal{O}(x,t)\mathcal{O}(0)|n\rangle \\ &= \sum_{n,m} e^{-(\beta-it)E_n - iE_m t}\langle n|\mathcal{O}(x,0)|m\rangle\langle m|\mathcal{O}(0)|n\rangle,\end{aligned} \tag{3.169}$$

then the correct prescription is to replace iy by $t - i\epsilon$, where ϵ is a small positive number.

Now consider the analytic continuation of the conformal block (3.164). We can set $z_2 = 0$ and $z_1 = x + iy$, and applying our prescription, replacing z_1 by $x + t - i\epsilon$. Similarly, we will analytically continue the complex conjugate, anti-holomorphic conformal block by sending $\bar{z}_1 \to x - t + i\epsilon$.

We are interested in the behavior of the two point function at time t of order $\mathcal{O}(N^0)$ but parametrically large. For this purpose, we may consider simply integer values of t and generic x. To obtain the values of the two-point function at integer time $t = n$, we can start at $(x, t = 0)$, and apply the t-monodromy which moves $t \to t + 1$ (with negative imaginary part so that $\mathcal{O}(x,t)$ goes below the insertion of $\mathcal{O}(0)$) n times. The t-monodromy on the holomorphic conformal block is given by

$$\mathcal{G}^t_{\lambda+\lambda'}(x + t + 1 + i\epsilon, 0|\tau) = e^{2\pi i\left(\sqrt{\frac{p'}{p}}P_N + \frac{p'(N-1)}{2pN}\right)}\mathcal{G}^t_{\lambda+\lambda'}(x + t + i\epsilon, 0|\tau). \tag{3.170}$$

The anti-holomorphic conformal block transforms with the same phase, due to the complex conjugation and the inverse t-monodromy. The phase factor is simply due to the difference of the conformal weight of the primary operators labeled by $u = \lambda + \lambda'$ and $u + \sqrt{\frac{p'}{p}}\mathbf{h}_k$ in

[16]If there is thermalization behavior at late time, the two-point function should decay in the distribution sense.

the t-channel. The two-point function at $t=n$ is then given by

$$\langle \mathcal{O}(x,t=n)\mathcal{O}(0)\rangle_\beta = \frac{1}{N!}\sum_{\lambda\in\Delta_1,\ \lambda'\in\Delta_2,\ w\in W} e^{2\pi i\left(2\sqrt{\frac{p'}{p}}w(P_N)+\frac{p'(N-1)}{pN}\right)}\left|\mathcal{C}_{w(u)}\mathcal{G}^t_{w(\lambda+\lambda')}(x|i\beta)\right|^2. \tag{3.171}$$

Recall that $w(P_N)=w(u)\cdot\mathbf{h}_N=u\cdot w^{-1}(\mathbf{h}_N)$, and $\sqrt{\frac{p'}{p}}w(P_N)$ is always an integer multiple of $1/(Np)$. So in fact the two-point function $\langle\mathcal{O}(x,t)\mathcal{O}(0)\rangle_\beta$ has time periodicity at most Np (this is simply a consequence of the fusion rule).

Unfortunately, we do not yet know a way to extract the large N behavior of the analytically continued two-point function, or even simply the two-point function at integer times, (3.171), for that matter. In the $N=2$ case, i.e. Virasoro minimal models,[17] the contour integral is one-dimensional, and we have computed (3.171) numerically in Appendix 3.F.

3.7 Conclusion

We have given in Section 3.4 the explicit formulae for the coefficients of all three-point functions of primaries in the W_N minimal model, subject to the condition that one of the primaries is of the form $(\otimes^n_{\rm sym}\bar{\mathbf{f}},\otimes^m_{\rm sym}\bar{\mathbf{f}})$, where $\otimes^n_{\rm sym}\bar{\mathbf{f}}$ is the n-th symmetric product tensor of the anti-fundamental representation $\bar{\mathbf{f}}$. This allows us to study the large N factorization and identify the bound state structure of a large class of operators. Apart form the elementary massive scalars $(\mathbf{f},0)=\phi$, $(0,\mathbf{f})=\widetilde{\phi}$, and the obvious elementary light state $(\mathbf{f},\mathbf{f})=\omega$, there are additional elementary light states e.g. $\frac{1}{\sqrt{2}}((S,S)-(A,A))$, as well as additional elementary massive states e.g. $\frac{1}{\sqrt{2}}((A,\mathbf{f})-(S,\mathbf{f}))=\Psi$. On the other hand, we have identified

[17] The contour integral expression for the torus two-point function in the Virasoro minimal model has been derived in [44]

the following operators as composite particles:

$$
\begin{aligned}
&(A,0) \sim \frac{1}{\sqrt{2}}\phi^2, \\
&(S,0) \sim \frac{1}{\sqrt{2}\Delta_{(\mathbf{f},0)}}(\phi\partial\bar{\partial}\phi - \partial\phi\bar{\partial}\phi), \\
&(adj,0) \sim \phi\bar{\phi}, \\
&\frac{(S,S)+(A,A)}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}\omega^2, \\
&\frac{(A,\mathbf{f})+(S,\mathbf{f})}{\sqrt{2}} \sim \phi\omega, \\
&\frac{(A,S)+(S,A)}{\sqrt{2}} \sim \frac{1}{\sqrt{2}\Delta_{(\mathbf{f},\mathbf{f})}}(\omega\partial\bar{\partial}\omega - \partial\omega\bar{\partial}\omega) \\
&\qquad\qquad\qquad \sim \frac{1}{\sqrt{2}}\left(\omega\phi\widetilde{\phi} - \frac{1}{\Delta_{(\mathbf{f},\mathbf{f})}}\partial\omega\bar{\partial}\omega\right), \\
&\frac{(A,S)-(S,A)}{\sqrt{2}} \sim \frac{\Psi\widetilde{\phi} - \widetilde{\Psi}\phi}{\sqrt{2}}.
\end{aligned}
\tag{3.172}
$$

We have also seen that the identification $\frac{1}{\Delta_{(\mathbf{f},\mathbf{f})}}\partial\bar{\partial}\omega \sim \phi\widetilde{\phi}$ of [12] is consistent with the large N factorization of composite operators. It would be nice to have a systematic classification of all elementary states/particles among the W_N primaries and their bound state structure. This should not be difficult using our approach.

The other main result of this paper is the exact torus two-point function of the basic primaries $(\mathbf{f},0)$ and $(\bar{\mathbf{f}},0)$, expressed explicitly as an $(N-1)$-fold contour integral. Direct evaluation of the contour integral appears difficult, but nonetheless feasible numerically at small N (as demonstrated in the $N=2$ case in Appendix 3.F). As our formulae are written for individual holomorphic conformal blocks, the analytic continuation to Lorentzian thermal two-point function is entirely straightforward. It would be very interesting to understand its large N behavior, say at time of order N^0. We expect some sort of thermalization behavior (as already shown in the $N=2$ example at large k, in fact) reflected in the decay of the two-point function in time, and the precise nature of the decay contains information about the dual bulk geometry. If the BTZ black hole dominates the thermodynamics at some temperature (above the Hawking-Page transition temperature), then we expect to see exponential decay of the thermal two-point function. To the best of our knowledge, such an exponential decay of the two-point function has not been demonstrated directly in a CFT with a semi-classical gravity dual (the closest being the long string CFT[18] of [46, 47] and in toy matrix quantum mechanics models [48, 49]). The W_N minimal model, being exactly solvable and has a weakly coupled gravity dual at large N (though seemingly very different from ordinary semi-classical gravity), seems to be a good place to address this issue. To extract the answer to this question from our result on the torus two-point function, however, is left to future work.

[18] The long string picture a priori holds in the orbifold point, which is far from the semi-classical regime in the bulk. One may expect that a similar qualitative picture holds for the deformed orbifold CFT in the semi-classical gravity regime, but showing this appears to be a nontrivial problem.

3.A The residues of Toda structure constants

Let us carry out the procedure of obtaining the structure constant $C_{W_N}(v_1, v_2, v_3)$ in the W_N minimal model by taking the residues of correlators in the affine Toda theory. Firstly, using (3.79), we derive the identities

$$\begin{aligned}&\frac{\Upsilon(x)}{\Upsilon(x+nb+m/b)}\\ &= (-1)^{mn}\left(\prod_{i=0}^{m-1}\prod_{j=0}^{n-1}\frac{1}{(i/b+x+jb)^2}\right)\left[\prod_{j=0}^{n-1}\frac{b^{-1+2bx+2jb^2}}{\gamma(bx+jb^2)}\right]\left[\prod_{j=0}^{m-1}\frac{b^{-2x/b-2j/b^2+1}}{\gamma(x/b+j/b^2)}\right],\end{aligned} \tag{3.173}$$

and

$$\begin{aligned}&\frac{\Upsilon(x)}{\Upsilon(x-nb-m/b)}\\ &= (-1)^{mn}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{(x-\frac{i}{b}-jb)^2}\right)\left[\prod_{j=1}^{n}\gamma(bx-jb^2)b^{1-2bx+2jb^2}\right]\left[\prod_{j=1}^{m}\gamma(x/b-j/b^2)b^{-1+2x/b-2j/b^2}\right].\end{aligned} \tag{3.174}$$

Next, we factorize the denominator of (3.76) into four groups, and substitute in (3.72), and set $\epsilon = 0$ in the factors that remains nonzero when $\epsilon = 0$. The factors in the denominator of (3.76) with $j > i$ become

$$\begin{aligned}&\Upsilon\Big(\frac{\varkappa}{N}+(\mathbf{v}_1-\mathcal{Q})\cdot\mathbf{h}_i+(\mathbf{v}_2-\mathcal{Q})\cdot\mathbf{h}_j\Big)\\ &= \Upsilon\Big(b(s_{i-1}-s_i)+\frac{1}{b}(s'_{i-1}-s'_i)+(\mathcal{Q}-\mathbf{v}_2)\cdot(\mathbf{h}_i-\mathbf{h}_j)\Big),\end{aligned} \tag{3.175}$$

and for $j < i$ we have

$$\begin{aligned}&\Upsilon\Big(\frac{\varkappa}{N}+(\mathbf{v}_1-\mathcal{Q})\cdot\mathbf{h}_i+(\mathbf{v}_2-\mathcal{Q})\cdot\mathbf{h}_j\Big)\\ &= \Upsilon\Big(b(s_{j-1}-s_j)+\frac{1}{b}(s'_{j-1}-s'_j)+(\mathcal{Q}-\mathbf{v}_1)\cdot(\mathbf{h}_j-\mathbf{h}_i)\Big).\end{aligned} \tag{3.176}$$

The denominator factors with $i = j = N$ become

$$\begin{aligned}&\Upsilon\Big(\frac{\varkappa}{N}+(\mathbf{v}_1-\mathcal{Q})\cdot\mathbf{h}_i+(\mathbf{v}_2-\mathcal{Q})\cdot\mathbf{h}_j\Big)\\ &= \Upsilon\Big(\varkappa+bs_{N-1}+\frac{1}{b}s'_{N-1}\Big),\end{aligned} \tag{3.177}$$

and for $i = j \neq N$, we have

$$\begin{aligned} &\Upsilon\Big(\frac{\varkappa}{N} + (\mathbf{v}_1 - \mathcal{Q})\cdot\mathbf{h}_j + (\mathbf{v}_2 - \mathcal{Q})\cdot\mathbf{h}_j + \epsilon\cdot\mathbf{h}_j\Big) \\ &= \Upsilon\Big(b(s_{j-1} - s_j) + \frac{1}{b}(s'_{j-1} - s'_j) + \epsilon_j - \epsilon_{j-1}\Big), \end{aligned} \tag{3.178}$$

where $s_0 = s'_0 = \epsilon_0 = 0$.

Now, it is clear that (3.178) are the only factors in the denominator that vanish at $\epsilon = 0$, and also they vanish only when $s_j \geq s_{j-1}$ and $s'_j \geq s'_{j-1}$, or $s_j < s_{j-1}$ and $s'_j < s'_{j-1}$. Let us first assume $s_j \geq s_{j-1}$ and $s'_j \geq s'_{j-1}$. We have

$$\begin{aligned} &\frac{\Upsilon(b)}{\Upsilon\Big(b(s_{j-1} - s_j) + \frac{1}{b}(s'_{j-1} - s'_j) + \epsilon\cdot\mathbf{h}_j\Big)} = \frac{1}{\epsilon\cdot\mathbf{h}_j}(-1)^{s'_{j,j-1}s_{j,j-1}}\left(\prod_{k=1}^{s'_{j,j-1}}\prod_{l=1}^{s_{j,j-1}}\frac{1}{(\epsilon\cdot\mathbf{h}_j + \frac{k}{b} + lb)^2}\right) \\ &\times\left[\prod_{l=1}^{s_{j,j-1}}\gamma(\epsilon\cdot\mathbf{h}_j - lb^2)\right]\left[\prod_{k=1}^{s'_{j,j-1}}\gamma(\epsilon\cdot\mathbf{h}_j - k/b^2)\right]\cdot b^{s_{j,j-1} - b^2(s_{j-1,j}-1)s_{j,j-1} - s'_{j,j-1} + b^2(s'_{j-1,j}-1)s'_{j,j-1}}, \end{aligned} \tag{3.179}$$

The prefactor $\frac{1}{\epsilon\cdot\mathbf{h}_j}$ is the only divergent piece in the $\epsilon \to 0$, and at this point we could take $\epsilon \to 0$ on the remaining factor, but we will keep the formula with nonzero ϵ for later use. There are also

$$\begin{aligned} &\frac{\Upsilon\Big((\mathcal{Q} - \mathbf{v}_2)\cdot(\mathbf{h}_j - \mathbf{h}_i)\Big)}{\Upsilon\Big(b(s_{j-1} - s_j) + \frac{1}{b}(s'_{j-1} - s'_j) + (\mathcal{Q} - \mathbf{v}_2)\cdot(\mathbf{h}_j - \mathbf{h}_i)\Big)} \\ &= (-1)^{s_{j,j-1}s'_{j,j-1}}\left(\prod_{k=1}^{s'_{j,j-1}}\prod_{l=1}^{s_{j,j-1}}\frac{1}{(\mathbf{P}^2_{ji} - \frac{k}{b} - lb)^2}\right)\times\left[\prod_{l=1}^{s_{j,j-1}}\gamma(b\mathbf{P}^2_{ji} - lb^2)\right]\left[\prod_{k=1}^{s'_{j,j-1}}\gamma(\mathbf{P}^2_{ji}/b - k/b^2)\right] \\ &\quad\times b^{s_{j,j-1} - b^2(s_{j-1,j}-1)s_{j,j-1} - s'_{j,j-1} + b^2(s'_{j-1,j}-1)s'_{j,j-1} - 2b\mathbf{P}^2_{ji}s_{j,j-1} + 2\mathbf{P}^2_{ji}s'_{j,j-1}/b}, \end{aligned} \tag{3.180}$$

and

$$\frac{\Upsilon\Big((\mathcal{Q}-\mathbf{v}_1)\cdot(\mathbf{h}_j-\mathbf{h}_i)\Big)}{\Upsilon\Big(b(s_{j-1}-s_j)+\frac{1}{b}(s'_{j-1}-s'_j)+(\mathcal{Q}-\mathbf{v}_1)\cdot(\mathbf{h}_j-\mathbf{h}_i)\Big)}$$
$$= (-1)^{s_{j,j-1}s'_{j,j-1}}\left(\prod_{k=1}^{s'_{j,j-1}}\prod_{l=1}^{s_{j,j-1}}\frac{1}{(\mathbf{P}^1_{ji}-\frac{k}{b}-lb)^2}\right)\times\left[\prod_{l=1}^{s_{j,j-1}}\gamma(b\mathbf{P}^1_{ji}-lb^2)\right]\left[\prod_{k=1}^{s'_{j,j-1}}\gamma(\mathbf{P}^1_{ji}/b-k/b^2)\right]$$
$$\times\, b^{s_{j,j-1}-b^2(s_{j-1,j}-1)s_{j,j-1}-s'_{j,j-1}+b^2(s'_{j-1,j}-1)s'_{j,j-1}-2b\mathbf{P}^1_{ji}s_{j,j-1}+2\mathbf{P}^1_{ji}s'_{j,j-1}/b}, \tag{3.181}$$

where we introduced the notation $s_{i,j}\equiv s_i-s_j$ and $\mathbf{P}^a_{ij}=(\mathcal{Q}-\mathbf{v}_a)\cdot(\mathbf{h}_i-\mathbf{h}_j)$, $a=1,2$. Combing the above three terms, we have

$$\frac{\Upsilon(b)}{\Upsilon\Big(bs_{j-1,j}+\frac{1}{b}s'_{j-1,j}+\epsilon\cdot\mathbf{h}_j\Big)}\prod_{i=j+1}^{N}\frac{\Upsilon\Big(\mathbf{P}^1_{ji}\Big)}{\Upsilon\Big(bs_{j-1,j}+\frac{1}{b}s'_{j-1,j}+\mathbf{P}^1_{ji}\Big)}\frac{\Upsilon\Big(\mathbf{P}^2_{ji}\Big)}{\Upsilon\Big(bs_{j-1,j}+\frac{1}{b}s'_{j-1,j}+\mathbf{P}^2_{ji}\Big)}$$
$$=\frac{1}{\epsilon\cdot\mathbf{h}_j}(-1)^{s'_{j,j-1}s_{j,j-1}}R^{s_{j,j-1},s'_{j,j-1}}_{j,\epsilon}b^{C_j}, \tag{3.182}$$

where $R^{s_{j,j-1},s'_{j,j-1}}_{j,\epsilon}$ is defined to be

$$\begin{aligned}R^{s_{j,j-1},s'_{j,j-1}}_{j,\epsilon} =& \left(\prod_{k=1}^{s'_{j,j-1}}\prod_{l=1}^{s_{j,j-1}}\frac{1}{(\epsilon\cdot\mathbf{h}_j+\frac{k}{b}+lb)^2}\prod_{i=j+1}^{N}\frac{1}{(\mathbf{P}^1_{ji}-\frac{k}{b}-lb)^2}\frac{1}{(\mathbf{P}^2_{ji}-\frac{k}{b}-lb)^2}\right)\\ &\times\left[\prod_{l=1}^{s_{j,j-1}}\gamma(\epsilon\cdot\mathbf{h}_j-lb^2)\prod_{i=j+1}^{N}\gamma(b\mathbf{P}^1_{ji}-lb^2)\gamma(b\mathbf{P}^2_{ji}-lb^2)\right]\\ &\times\left[\prod_{k=1}^{s'_{j,j-1}}\gamma(\epsilon\cdot\mathbf{h}_j-k/b^2)\prod_{i=j+1}^{N}\gamma(\mathbf{P}^1_{ji}/b-k/b^2)\gamma(\mathbf{P}^2_{ji}/b-k/b^2)\right].\end{aligned} \tag{3.183}$$

The exponent C_j of b is given by

$$\begin{aligned}C_j =& (2N-2j+1)(s_{j,j-1}-s'_{j,j-1})+2(s_{j-1}s'_j-s_js'_{j-1})+b^2\left[(2N-2j+1)s_{j,j-1}+s^2_{j-1}-s^2_j\right]\\ &-\frac{1}{b^2}\left[(2N-2j+1)s'_{j,j-1}+s'^2_{j-1}-s'^2_j\right]-2bs_{j,j-1}\varkappa+2\frac{1}{b}s'_{j,j-1}\varkappa,\end{aligned} \tag{3.184}$$

where we have used

$$\sum_{i=j+1}^{N}(\mathbf{P}^1_{ji}+\mathbf{P}^2_{ji}) \quad = \varkappa + b(N-j)s_{j,j-1} + bs_j + \frac{1}{b}(N-j)s'_{j,j-1} + \frac{1}{b}s'_j. \tag{3.185}$$

We also have

$$\begin{aligned}\frac{\Upsilon(\varkappa)}{\Upsilon(\varkappa+bs_{N-1}+s'_{N-1}/b)} &= (-1)^{s_{N-1}s'_{N-1}}\left(\prod_{k=0}^{s'_{N-1}-1}\prod_{l=0}^{s_{N-1}-1}\frac{1}{(\varkappa+\frac{k}{b}+lb)^2}\right)\\ &\times\left[\prod_{l=0}^{s_{N-1}-1}\gamma(1-b\varkappa-lb^2)\right]\left[\prod_{k=0}^{s'_{N-1}-1}\gamma(1-\varkappa/b-k/b^2)\right]\\ &\times b^{-s_{N-1}+2b\varkappa s_{N-1}+b^2s_{N-1}(s_{N-1}-1)+s'_{N-1}-2\frac{1}{b}\varkappa s'_{N-1}-\frac{1}{b^2}s'_{N-1}(s'_{N-1}-1)}.\end{aligned} \tag{3.186}$$

Putting the above terms together, the total exponent of b is

$$\begin{aligned}&\sum_{j=1}^{N-1}C_j - s_{N-1}+2b\varkappa s_{N-1}+b^2s_{N-1}(s_{N-1}-1)+s'_{N-1}-2\frac{1}{b}\varkappa s'_{N-1}-\frac{1}{b^2}s'_{N-1}(s'_{N-1}-1)\\ &=2(1+b^2)\sum_{j=1}^{N-1}s_j-2(1+\frac{1}{b^2})\sum_{j=1}^{N-1}s'_j+2\sum_{j=1}^{N-2}(s_js'_{j+1}-s_{j+1}s'_j).\end{aligned} \tag{3.187}$$

Finally, we rewrite the prefactor of (3.76) in the form

$$\left[\mu\pi\gamma(b^2)b^{2-2b^2}\right]^{\frac{(2Q-\sum\mathbf{v}_i,\rho)}{b}} \quad = \left[\frac{-\mu\pi}{\gamma(-b^2)}b^{-2-2b^2}\right]^{\sum_{k=1}^{N-1}s_k}\left[\frac{-\mu'\pi}{\gamma(-\frac{1}{b^2})}b^{\frac{2}{b^2}+2}\right]^{\sum_{k=1}^{N-1}s'_k}. \tag{3.188}$$

The residue of the three point function is then

$$\begin{aligned}&\mathrm{res}_{\epsilon_1\to0}\mathrm{res}_{\epsilon_2\to\epsilon_1}\cdots\mathrm{res}_{\epsilon_{N-1}\to\epsilon_{N-2}}C_{toda}(\mathbf{v}_1,\mathbf{v}_2,\varkappa\omega_{n-1})\\ &=(ib)^{2\sum_{j=1}^{N-2}(s_js'_{j+1}-s_{j+1}s'_j)}\left[\frac{-\mu\pi}{\gamma(-b^2)}\right]^{\sum_{k=1}^{N-1}s_k}\left[\frac{-\mu'\pi}{\gamma(-\frac{1}{b^2})}\right]^{\sum_{k=1}^{N-1}s'_k}\left(\prod_{k=0}^{s'_{N-1}-1}\prod_{l=0}^{s_{N-1}-1}\frac{1}{(\varkappa+\frac{k}{b}+lb)^2}\right)\\ &\times\left[\prod_{l=0}^{s_{N-1}-1}\gamma(1-b\varkappa-lb^2)\right]\left[\prod_{k=0}^{s'_{N-1}-1}\gamma(1-\varkappa/b-k/b^2)\right]\prod_{j=1}^{N-1}R_{j,\epsilon}^{s_{j,j-1},s'_{j,j-1}}.\end{aligned} \tag{3.189}$$

The ϵ is the subscript of $R^{s,s'}_{j,\epsilon}$ is understand to be taken to zero in computing the residue, but we will leave it in the formula as we will make use of it below.

In the case $s_j < s_{j-1}$ and $s'_j < s'_{j-1}$, we can apply the following identity:

$$\frac{\Upsilon(x)}{\Upsilon(x-nb-m/b)} = \frac{\Upsilon(b+1/b-x)}{\Upsilon(b+1/b-x+nb+m/b)}, \tag{3.190}$$

and then the residue will be computed by the above formula with the replacement

$$\epsilon \to b+1/b-\epsilon, \quad \mathbf{P}^1_{ji} \to b+1/b-\mathbf{P}^1_{ji}, \quad \mathbf{P}^2_{ji} \to b+1/b-\mathbf{P}^2_{ji}, \tag{3.191}$$

and then set ϵ to zero. Finally, we obtain the structure constants in the W_N minimal model by the analytic continuation (3.74).

3.B Monodromy of integration contours

In this appendix, we analyze the s and t channel monodromy action on the contour integrals described in Section 3.5.2.

Let us begin by considering the s_2-integral. The s_2-integrand has branch points at $0, s_1, x_2, \infty$. There are relations among the L contours encircling a pair of the branch points. For instance,

$$\begin{aligned}
&\int_{L(0,\infty)} = -\int_{L(0,\{s_1,x_2\})} \\
&= \int_0^{s_1} + \int_{s_1}^{x_2} + g_{x_2}\int_{x_2}^{s_1} + g_{x_2}g_{s_1}\int_{s_1}^{0} + g_{x_2}g_{s_1}g_0\int_0^{s_1} + g_{x_2}g_{s_1}g_0g_{s_1}^{-1}\int_{s_1}^{x_2} + g_{x_2}g_{s_1}g_0g_{s_1}^{-1}g_{x_2}^{-1}\left(\int_{x_2}^{s_1} + \int_{s_1}^{0}\right) \\
&= (1-g_{x_2}g_{s_1}+g_{x_2}g_{s_1}g_0-g_{x_2}g_{s_1}g_0g_{s_1}^{-1}g_{x_2}^{-1})\int_0^{s_1} + (1-g_{x_2}+g_{x_2}g_{s_1}g_0g_{s_1}^{-1}-g_{x_2}g_{s_1}g_0g_{s_1}^{-1}g_{x_2}^{-1})\int_{s_1}^{x_2}.
\end{aligned} \tag{3.192}$$

Now since all the g's are commuting phase factors, we can write

$$\int_{L(0,\infty)} = (1-g_0)(1-g_{s_1}g_{x_2})\int_0^{s_1} + (1-g_0)(1-g_{x_2})\int_{s_1}^{x_2}. \tag{3.193}$$

Naively, one may think that the integral over $L(x_2, \infty)$ is given by the same expression with 0 and x_2 exchanged. This is not correct, however, due to the choice of branch in the line integrals. We have

$$\begin{aligned}
\int_{L(x_2,\infty)} &= -\int_{L(x_2,\{s_1,x_0\})} \\
&= \int_{x_2}^{s_1} + g_{s_1}\int_{s_1}^{0} + g_{s_1}g_0\left(\int_0^{s_1} + \int_{s_1}^{x_2}\right) + g_{s_1}g_0g_{x_2}\left(\int_{x_2}^{s_1} + \int_{s_1}^{0}\right) + g_{s_1}g_0g_{x_2}g_0^{-1}\int_0^{s_1} + g_{s_1}g_0g_{x_2}g_0^{-1}g_{s_1}^{-1}\int_{s_1}^{x_2} \\
&= -(1-g_{s_1}g_0)(1-g_{x_2})\int_{s_1}^{x_2} - g_{s_1}(1-g_0)(1-g_{x_2})\int_0^{s_1}.
\end{aligned} \tag{3.194}$$

Together with using the following relation between the L-contour and the "collapsed" line integral,

$$\begin{aligned}
\int_{L(0,s_1)} &= (1-g_{s_1})(1-g_0)\int_0^{s_1}, \\
\int_{L(s_1,x_2)} &= (1-g_{s_1})(1-g_{x_2})\int_{s_1}^{x_2},
\end{aligned} \tag{3.195}$$

we derive the formula (3.125).

Now consider the two-dimensional contours (3.126). Let us denote by $\mathcal{I}^{(i)}$ the contours obtained from $C^{(i)}$ by collapsing $L(z_1, z_2)$ into straight lines, and by J_i the integral of (3.122) along $\mathcal{I}^{(i)}$, and also by $\mathcal{J}_i$ the integral of (3.122) along $C^{(i)}$, $i = 1, 2, 3, 4$. J_i and $\mathcal{J}_i$ are related via

$$\begin{aligned}
\begin{pmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \end{pmatrix} &= (\mathbf{1} - g_{x_2}(s_1))(\mathbf{1} - g_{x_1}(s_1)) \begin{pmatrix} (1-g_{x_2})(1-g_\infty)J_1 \\ (1-g_0)(1-g_{s_1})J_2 \end{pmatrix}, \\
\begin{pmatrix} \mathcal{J}_3 \\ \mathcal{J}_4 \end{pmatrix} &= (\mathbf{1} - g_0(s_1))(\mathbf{1} - g_{x_1}(s_1)) \begin{pmatrix} (1-g_{x_2})(1-g_\infty)J_3 \\ (1-g_0)(1-g_{s_1})J_4 \end{pmatrix}.
\end{aligned} \tag{3.196}$$

T_t and T_s acts on $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4)$ via the monodromy matrices (3.127) and (3.128). Define $\zeta \equiv e^{2\pi i \frac{p'}{p}}$. We find

$$g_0(x_1) = \zeta^{\frac{2}{3}n_+ + \frac{1}{3}m_+} e^{-2\pi i(\frac{2}{3}n_- + \frac{1}{3}m_-)}, \qquad g_{x_2}(x_1) = \zeta^{\frac{1}{3}}, \tag{3.197}$$

and

$$g_0(s_1) = \begin{pmatrix} \zeta^{-n_+} & 0 \\ 0 & \zeta^{-n_+-m_+-1} \end{pmatrix}, \quad g_{x_1}(s_1) = \zeta^{-1}\mathbf{1}, \quad g_{x_2}(s_1) = A^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \zeta^{-2} \end{pmatrix} A. \quad (3.198)$$

The matrix A is the linear transformation of L-contours,

$$\begin{pmatrix} L(0,\infty) \\ L(s_1,x_2) \end{pmatrix} = A \begin{pmatrix} L(x_2,\infty) \\ L(0,s_1) \end{pmatrix}, \tag{3.199}$$

and from (3.125) we know

$$A = -\frac{1}{1-g_0 g_{s_1}} \begin{pmatrix} 1-g_0 & -1+g_0 g_{x_2} g_{s_1} \\ 1-g_{s_1} & g_{s_1}(1-g_{x_2}) \end{pmatrix}. \tag{3.200}$$

Using the monodromy phases of the s_2-integrand,

$$g_0 = \zeta^{-m_+}, \quad g_{s_1} = g_{x_2} = \zeta^{-1}, \tag{3.201}$$

we find

$$A = -\frac{1}{1-\zeta^{-m_+-1}} \begin{pmatrix} 1-\zeta^{-m_+} & \zeta^{-m_+-2}-1 \\ 1-\zeta^{-1} & \zeta^{-1}-\zeta^{-2} \end{pmatrix}. \tag{3.202}$$

3.C Identifying the conformal blocks with contour integrals

It is useful to work in instead of $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4)$, the basis

$$\begin{aligned} \begin{pmatrix} \widetilde{\mathcal{J}}_1 \\ \widetilde{\mathcal{J}}_2 \end{pmatrix} &= A \begin{pmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \end{pmatrix} = \int_{L(x_1,x_2)} ds_1 \begin{pmatrix} \int_{L(0,\infty)} ds_2 \cdots \\ \int_{L(s_1,x_2)} ds_2 \cdots \end{pmatrix}, \\ \begin{pmatrix} \mathcal{J}_3 \\ \mathcal{J}_4 \end{pmatrix} &= \int_{L(0,x_1)} ds_1 \begin{pmatrix} \int_{L(x_2,\infty)} ds_2 \cdots \\ \int_{L(0,s_1)} ds_2 \cdots \end{pmatrix}. \end{aligned} \tag{3.203}$$

In fact, $\widetilde{\mathcal{J}}_1$ vanishes identically, as a consequence of the relation

$$A(\mathbf{1}-g_{x_2}(s_1))(\mathbf{1}-g_{x_1}(s_1)) = -\frac{\zeta^{4-m_+}(1-\zeta^{1+m_+})}{(1+\zeta)(1-\zeta)^3}\begin{pmatrix} 0 & 0 \\ \zeta & 1 \end{pmatrix}. \tag{3.204}$$

Acting on $(\widetilde{\mathcal{J}}_2, \mathcal{J}_3, \mathcal{J}_4)$, the monodromy matrices are of the form

$$\begin{aligned} \tilde{M}_s &= \zeta^{\frac{1}{3}}\begin{pmatrix} \zeta^{-3} & 0 & 0 \\ \frac{(1-\zeta^{2+m_+})(1-\zeta^{n_+})}{\zeta^{1+m_++n_+}(1-\zeta^2)} & 1 & 0 \\ \frac{(1-\zeta^{m_+})(1-\zeta^{1+m_++n_+})}{\zeta^{2m_++n_+}(1-\zeta^2)} & 0 & 1 \end{pmatrix}, \\ \tilde{M}_t &= \zeta^{\frac{2}{3}n_++\frac{1}{3}m_+}e^{-2\pi i(\frac{2}{3}n_-+\frac{1}{3}m_-)}\begin{pmatrix} 1 & \frac{(1-\zeta)^2(1+\zeta)\zeta^{-2+m_+}}{1-\zeta^{m_++1}} & \frac{(1-\zeta)^2(1+\zeta)\zeta^{-3+m_+}}{1-\zeta^{m_++1}} \\ 0 & \zeta^{-1-n_+} & 0 \\ 0 & 0 & \zeta^{-2-n_+-m_+} \end{pmatrix}. \end{aligned} \tag{3.205}$$

As described in Section 3.5.3, the four point function is obtained by summing over either s or t channel conformal blocks (3.136). The mass matrices therein, $\mathcal{M}^t$ and $\mathcal{M}^s$, are of the form

$$\mathcal{M}^t = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \mathcal{M}^s = \begin{pmatrix} d & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \tag{3.206}$$

and obey (3.137). (3.137) is solved with

$$\begin{aligned} \frac{a}{c} &= \frac{\zeta^{2-2m_+-n_+}(1-\zeta^{m_+})(1-\zeta^{1+m_+})(1-\zeta^{1+n_+})(1-\zeta^{1+m_++n_+})(1-\zeta^{2+m_++n_+})^2}{(1-\zeta)^4(1+\zeta)^2(1-\zeta^{2+n_+})(1-\zeta^{3+m_++n_+})}, \\ \frac{b}{c} &= \frac{\zeta^{-m_+}(1-\zeta^{m_+})(1-\zeta^{1+m_++n_+})(1-\zeta^{2+m_++n_+})}{(1-\zeta^{2+m_+})(1-\zeta^{n_+})(1-\zeta^{1+n_+})}. \end{aligned} \tag{3.207}$$

The overall normalization can be fix by the identity s-channel, which then fixes the entire four point function. From this four point function one may also extract the coefficients of the sphere 3-point functions, $\langle \mathcal{O}_{(adj,0)}\mathcal{O}_u\overline{\mathcal{O}_u}\rangle$, $\langle \mathcal{O}_{(adj',0)}\mathcal{O}_u\overline{\mathcal{O}_u}\rangle$, etc., and reproduce some of the results in Section 3.3.

3.D Monodromy invariance of the sphere four-point function

In this section, we show that the formula (3.148) for the four-point function is invariant under the t- and u-monodromy transformations, i.e. circling x_1 around 0 and ∞. By (3.144), the t-monodromy acting as a phase on the t-channel conformal blocks; hence, the the four-point function (3.148) is trivially invariant. To exhibit the u-monodromy, let us apply the following identity on the generalized hypergeometric function:

$$\begin{aligned}
&{}_N F_{N-1}(a_1,\cdots,a_N;b_1,\cdots,b_{N-1}|x)\\
&=\frac{\prod_{k=1}^{N-1}\Gamma(b_k)}{\prod_{k=1}^{N}\Gamma(a_k)}\sum_{k=1}^{N}\frac{\Gamma(a_k)\prod_{j=1,j\neq k}^{N}\Gamma(a_j-a_k)}{\prod_{j=1}^{N-1}\Gamma(b_j-a_k)}(-x)^{-a_k}\\
&\quad\times{}_N F_{N-1}(a_k,a_k-b_1+1,\cdots,a_k-b_{N-1}+1;1-a_1+a_k,\cdots,1-a_N+a_k|\frac{1}{x}).
\end{aligned}\tag{3.208}$$

Via this identity, the conformal block $G_l(x)$ can be rewrited as

$$\begin{aligned}
G_l(x)&=x^{\sqrt{\frac{p'}{p}}P_l}{}_N F_{N-1}(\vec{\mu}_l;\widehat{\vec{\nu}}_l|x)\\
&=\frac{\prod_{i=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{li}+1)}{\prod_{i=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{li}+\frac{p'}{p})}\sum_{k=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{lk}+\frac{p'}{p})\Gamma(1+\sqrt{\frac{p'}{p}}P_{kl}-\frac{p'}{p})\\
&\quad\times\frac{\prod_{j=1,j\neq k}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{kj})}{\prod_{j=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{kj}+1-\frac{p'}{p})}e^{i\pi\left(\sqrt{\frac{p'}{p}}P_{kl}-\frac{p'}{p}\right)}H_k(x),
\end{aligned}\tag{3.209}$$

where $H_k(x)$ are the u-channel conformal blocks, given by

$$H_k(x)=x^{\sqrt{\frac{p'}{p}}P_k-\frac{p'}{p}}{}_N F_{N-1}(\vec{\mu}'_k;\widehat{\vec{\nu}}'_k|\frac{1}{x}),\tag{3.210}$$

and $\vec{\mu}'_k$, $\vec{\nu}'_k$ are N-vectors defined as

$$\begin{aligned}\vec{\mu}'_k &= \sqrt{\frac{p'}{p}}(P_{1,k}, \cdots, P_{N,k}) + \frac{p'}{p}(1, \cdots, 1),\\ \vec{\nu}'_k &= \sqrt{\frac{p'}{p}}(P_{1,k}, \cdots, P_{N,k}) + (1, \cdots, 1).\end{aligned} \tag{3.211}$$

Again $\widehat{\vec{\nu}}'_k$ is given by $\vec{\nu}'_k$ dropping the k-th entry. In terms of the u-channel conformal blocks $H_l(x)$, the four-point function can be written as

$$\begin{aligned}&\sum_{l=1}^{N}(\mathcal{M}_u)_{ll}|G_l(x)|^2\\ &= \gamma\left(\frac{p'}{p}\right)\gamma\left(N\left(1-\frac{p'}{p}\right)\right)\sum_{l=1}^{N}\left(\prod_{i=1,i\neq l}^{N}\frac{\Gamma(\sqrt{\frac{p'}{p}}P_{il})\Gamma(1-\sqrt{\frac{p'}{p}}P_{il})}{\Gamma(1-\frac{p'}{p}+\sqrt{\frac{p'}{p}}P_{il})\Gamma(\frac{p'}{p}-\sqrt{\frac{p'}{p}}P_{il})}\right)\frac{1}{\Gamma(\frac{p'}{p})^2}\\ &\quad\times\sum_{k_1=1}^{N}\sum_{k_2=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{lk_1}+\frac{p'}{p})\Gamma(1-\sqrt{\frac{p'}{p}}P_{lk_1}-\frac{p'}{p})\Gamma(\sqrt{\frac{p'}{p}}P_{lk_2}+\frac{p'}{p})\Gamma(1-\sqrt{\frac{p'}{p}}P_{lk_2}-\frac{p'}{p})e^{i\pi\sqrt{\frac{p'}{p}}P_{k_1k_2}}\\ &\quad\times\frac{\prod_{j=1,j\neq k_1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{k_1j})}{\prod_{j=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{k_1j}+1-\frac{p'}{p})}\frac{\prod_{j=1,j\neq k_2}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{k_2j})}{\prod_{j=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{k_2j}+1-\frac{p'}{p})}H_{k_1}(x)H_{k_2}(\bar{x}).\end{aligned} \tag{3.212}$$

Using the following identity

$$\begin{aligned}&\sum_{l=1}^{N}\left(\prod_{i=1,i\neq l}^{N}\frac{\Gamma(\sqrt{\frac{p'}{p}}P_{il})\Gamma(1-\sqrt{\frac{p'}{p}}P_{il})}{\Gamma(1-\frac{p'}{p}+\sqrt{\frac{p'}{p}}P_{il})\Gamma(\frac{p'}{p}-\sqrt{\frac{p'}{p}}P_{il})}\right)\\ &\times\Gamma(\sqrt{\frac{p'}{p}}P_{lk_1}+\frac{p'}{p})\Gamma(1-\sqrt{\frac{p'}{p}}P_{lk_1}-\frac{p'}{p})\Gamma(\sqrt{\frac{p'}{p}}P_{lk_2}+\frac{p'}{p})\Gamma(1-\sqrt{\frac{p'}{p}}P_{lk_2}-\frac{p'}{p})e^{i\pi\sqrt{\frac{p'}{p}}P_{k_1k_2}}\\ &= \pi^2\sum_{l=1}^{N}\left(\prod_{i=1,i\neq l}^{N}\frac{\sin\pi(\frac{p'}{p}-\sqrt{\frac{p'}{p}}P_{il})}{\sin\pi(\sqrt{\frac{p'}{p}}P_{il})}\right)\csc\pi(\sqrt{\frac{p'}{p}}P_{lk_1}+\frac{p'}{p})\csc\pi(\sqrt{\frac{p'}{p}}P_{lk_2}+\frac{p'}{p})e^{i\pi\sqrt{\frac{p'}{p}}P_{k_1k_2}}\\ &\propto\delta_{k_1,k_2},\end{aligned} \tag{3.213}$$

(3.212) may be simplified to

$$\begin{aligned}
&\sum_{l=1}^{N}(\mathcal{M}_u)_{ll}|G_l(x)|^2 \\
&= \gamma\left(\frac{p'}{p}\right)\gamma\left(N\left(1-\frac{p'}{p}\right)\right)\sum_{k=1}^{N}\sum_{l=1}^{N}\left(\prod_{i=1,i\neq l}^{N}\frac{\Gamma(\sqrt{\frac{p'}{p}}P_{il})\Gamma(1-\sqrt{\frac{p'}{p}}P_{il})}{\Gamma(1-\frac{p'}{p}+\sqrt{\frac{p'}{p}}P_{il})\Gamma(\frac{p'}{p}-\sqrt{\frac{p'}{p}}P_{il})}\right)\frac{1}{\Gamma(\frac{p'}{p})^2} \\
&\quad\times\Gamma(\sqrt{\frac{p'}{p}}P_{lk}+\frac{p'}{p})^2\Gamma(1-\sqrt{\frac{p'}{p}}P_{lk}-\frac{p'}{p})^2\left(\prod_{j=1,j\neq k}^{N}\frac{\Gamma(\sqrt{\frac{p'}{p}}P_{kj})}{\Gamma(\sqrt{\frac{p'}{p}}P_{kj}+1-\frac{p'}{p})}\right)^2\frac{1}{\Gamma(1-\frac{p'}{p})^2} \\
&\quad\times|H_k(x)|^2 \\
&= \gamma\left(\frac{p'}{p}\right)\gamma\left(N\left(1-\frac{p'}{p}\right)\right)\sum_{j=1}^{N}\prod_{i=1,i\neq j}^{N}\frac{\Gamma(\sqrt{\frac{p'}{p}}P_{ji})\Gamma(\frac{p'}{p}-\sqrt{\frac{p'}{p}}P_{ji})}{\Gamma(1-\frac{p'}{p}+\sqrt{\frac{p'}{p}}P_{ji})\Gamma(1-\sqrt{\frac{p'}{p}}P_{ji})}|H_j(x)|^2 \\
&= \sum_{j=1}^{N}(\widetilde{\mathcal{M}}_u)_{jj}|H_j(x)|^2,
\end{aligned} \tag{3.214}$$

where the u-channel mass matrix $\widetilde{\mathcal{M}}_u$ is given in terms of the structure constants as (here the subscript u is the charge vector)

$$\begin{aligned}
(\widetilde{\mathcal{M}}_u)_{jj} = B\left(\sqrt{\frac{p'}{p}}w^1\right)^2 C_{W_N}\left(\sqrt{\frac{p'}{p}}w^1, Q-u, Q+u-\sqrt{\frac{p'}{p}}\mathbf{h}_j\right) \\
\times C_{W_N}\left(Q-u+\sqrt{\frac{p'}{p}}\mathbf{h}_j, \sqrt{\frac{p'}{p}}w^{N-1}, u+Q\right)
\end{aligned} \tag{3.215}$$

The u-monodromy acts as a phase on the u-channel conformal blocks (3.210). The four-point function (3.214) is invariant.

3.E q-expansion of the torus two-point function

In this section, we study the q-expansion of the torus conformal block (3.164). Let us start by expanding (3.162) as

$$G_u^{bos}(z_1,z_2|\tau) = \sum_{n\in\Gamma_{pp'}} q^{-\frac{N-1}{24}+\frac{1}{2}(u+n)^2}\left[G_{u+n}^{bos,(0)}(z_1,z_2) + G_{u+n}^{bos,(1)}(z_1,z_2)q + \mathcal{O}(q^2)\right], \tag{3.216}$$

where $G_u^{bos,(n)}(z_1,z_2)$ are obtained from the q-expansion of the θ_1 and η functions in (3.162). For simplicity, here we will assume that N is sufficiently large, and examine only the first few terms in the q expansion. For this purpose, we can ignore the sum over the lattice $\Gamma_{pp'}$ by setting $n=0$, while restricting $u\in\Gamma_{pp'}^*/\Gamma_{pp'}$ to take the value in the equivalence class that minimize u^2, since the effects of nonzero n only come in of the order $q^{\sim N^2}$. Plugging this formula into (3.163) and (3.164), we obtain

$$\mathcal{G}^t_{\lambda+\lambda'}(z_1,z_2|\tau) = \sum_{w\in W} q^{-\frac{N-1}{24}+\frac{1}{2}(\lambda+w(\lambda'))^2}\left[\mathbf{G}^{(0)}_{\lambda+w(\lambda')}(z_1,z_2) + \mathbf{G}^{(1)}_{\lambda+w(\lambda')}(z_1,z_2)q + \mathcal{O}(q^2)\right]. \tag{3.217}$$

Next, we expand the product of theta functions in (3.162),

$$\begin{aligned}
&\frac{1}{\eta(\tau)^{N-1}}\left(\frac{\theta_1(z_{12}|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{\frac{p'}{pN}}\left(\frac{\theta_1(z_1-t_1|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{-\frac{p'}{p}}\left(\frac{\theta_1(z_2-t_{N-1}|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{-\frac{p'}{p}}\prod_{i=1}^{N-2}\left(\frac{\theta_1(t_{i,i+1}|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{-\frac{p'}{p}}\\
&= q^{-\frac{N-1}{24}}\left(\frac{i}{4\pi}\right)^{\frac{p'}{pN}-\frac{p'}{p}N}\left(\frac{x_{12}}{\sqrt{x_1x_2}}\right)^{\frac{p'}{pN}}\left(\frac{x_1-s_1}{\sqrt{x_1s_1}}\right)^{-\frac{p'}{p}}\left(\frac{x_2-s_{N-1}}{\sqrt{s_{N-1}x_2}}\right)^{-\frac{p'}{p}}\prod_{i=1}^{N-2}\left(\frac{s_{i,i+1}}{\sqrt{s_is_{i+1}}}\right)^{-\frac{p'}{p}}\\
&\quad\times\left[1+\left(N-1-\frac{p'}{pN}\frac{x_{12}^2}{x_1x_2}\right)q + \frac{p'}{p}\left(\sum_{k=1}^{N-1}\frac{s_{k-1,k}^2}{s_{k-1}s_k}+\frac{(x_2-s_{N-1})^2}{s_{N-1}x_2}\right)q+\mathcal{O}(q^2)\right],
\end{aligned} \tag{3.218}$$

where $s_0 \equiv x_1$, and we have made a conformal transformation $x_i = e^{2\pi i z_i}$ and $s_i = e^{2\pi i t_i}$. The zeroth order term in this expansion, after the contour integral, gives[19]

$$\mathbf{G}_u^{(0)}(z_1, z_2) = \left(\frac{i}{4\pi}\right)^{\frac{p'}{pN} - \frac{p'}{p}N} (x_2 - x_1)^{\frac{p'}{Np}} x_1^{\frac{p'(N-1)}{2pN}} x_2^{\frac{p'(N-1)}{2pN} - \frac{p'}{p}} \mathbf{G}_u\left(\frac{x_1}{x_2}\right). \tag{3.219}$$

The first order terms in the expansion (3.217) can be split into three terms,

$$\mathbf{G}_u^{(1)}(z_1, z_2) = \mathbf{G}_u^{(1),1}(z_1, z_2) + \mathbf{G}_u^{(1),2}(z_1, z_2) + \mathbf{G}_u^{(1),3}(z_1, z_2), \tag{3.220}$$

coming from the three terms of order q in the second line of (3.218),

$$\left(N - 1 + \frac{p'}{pN}\frac{x_{12}^2}{x_1 x_2}\right), \quad \frac{p'}{p}\sum_{k=1}^{N-1} \frac{s_{k-1,k}^2}{s_{k-1}s_k}, \quad \frac{p'}{p}\sum_{k=1}^{N-1} \frac{(x_2 - s_{N-1})^2}{s_{N-1}x_2}. \tag{3.221}$$

The first term is independent of s_i and its contribution is proportional to $\mathbf{G}_u^{(0)}$ after doing the contour integral. The second term of (3.220) is computed as

$$\begin{aligned}
\mathbf{G}_u^{(1),2}(z_1, z_2) &= \left(\frac{i}{4\pi}\right)^{\frac{p'}{pN} - \frac{p'}{p}N} \frac{p'}{p}\sum_{k=1}^{N-1} \frac{s_{k-1,k}^2}{s_{k-1}s_k}(x_2 - x_1)^{\frac{p'}{Np}} x_1^{\sqrt{\frac{p'}{p}}P_1 + \frac{p'(N-1)}{2pN}} x_2^{-\sqrt{\frac{p'}{p}}P_N + \frac{p'(N-1)}{2pN}} \\
&\times \int_0^{x_1} ds_1\; s_1^{-\sqrt{\frac{p'}{p}}(u+Q)\cdot\alpha_1} (x_1 - s_1)^{-\frac{p'}{p}} \\
&\times \left(\prod_{i=1}^{N-2}\int_0^{s_i} ds_{i+1}\; s_{i+1}^{-\sqrt{\frac{p'}{p}}(u+Q)\cdot\alpha_{i+1}} (s_i - s_{i+1})^{-\frac{p'}{p}}\right)(x_2 - s_{N-1})^{-\frac{p'}{p}} \\
&= \left(\frac{i}{4\pi}\right)^{\frac{p'}{pN} - \frac{p'}{p}N} (x_2 - x_1)^{\frac{p'}{Np}} x_1^{\sqrt{\frac{p'}{p}}P_N + \frac{p'(N-1)}{2pN}} x_2^{-\sqrt{\frac{p'}{p}}P_N + \frac{p'(N-1)}{2pN} - \frac{p'}{p}} \\
&\times \frac{p'}{p}\sum_{k=1}^{N-1} \frac{\prod_{i=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{N,i} + \frac{p'}{p} - \delta_{i,k})}{\prod_{i=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{N,i} + 1 + \delta_{i,k})}\frac{\Gamma(1 - \frac{p'}{p})^{N-1}}{\Gamma(\frac{p'}{p})}(1 - \frac{p'}{p})(2 - \frac{p'}{p}) \\
&\times {}_NF_{N-1}(\vec{\mu}_N - \delta_k; \widehat{\vec{\nu}}_N + \delta_k | \frac{x_1}{x_2}),
\end{aligned} \tag{3.222}$$

[19]Here a conformal factor of the form $x_1^{h_{\mathbf{f}}} x_2^{h_{\bar{\mathbf{f}}}}$, together with the factors in (3.147), is included in rewriting $\mathbf{G}_u^{(0)}$ in terms of the sphere four-point conformal block $\mathbf{G}_u$.

where $\vec{1} = (1, \cdots, 1)$ and $(\vec{\delta}_k)_i = \delta_{k,i}$. The third term of (3.220) is given by

$$\begin{aligned}
\mathbf{G}_u^{(1),3}(z_1, z_2) &= \left(\frac{i}{4\pi}\right)^{\frac{p'}{pN}-\frac{p'}{p}N} \frac{p'}{p}\frac{(s_{N-1}-x_2)^2}{x_2 s_{N-1}}(x_2-x_1)^{\frac{p'}{Np}} x_1^{\sqrt{\frac{p'}{p}}P_1+\frac{p'(N-1)}{2pN}} x_2^{-\sqrt{\frac{p'}{p}}P_N+\frac{p'(N-1)}{2pN}} \\
&\times \int_0^{x_1} ds_1\, s_1^{-\sqrt{\frac{p'}{p}}(u+Q)\cdot\alpha_1}(x_1-s_1)^{-\frac{p'}{p}} \left(\prod_{i=1}^{N-2}\int_0^{s_i} ds_{i+1}\, s_{i+1}^{-\sqrt{\frac{p'}{p}}(u+Q)\cdot\alpha_{i+1}}(s_i-s_{i+1})^{-\frac{p'}{p}}\right)(x_2-s_{N-1})^{-\frac{p'}{p}} \\
&= \left(\frac{i}{4\pi}\right)^{\frac{p'}{pN}-\frac{p'}{p}N} (x_2-x_1)^{\frac{p'}{Np}} x_1^{\sqrt{\frac{p'}{p}}P_N+\frac{p'(N-1)}{2pN}-1} x_2^{-\sqrt{\frac{p'}{p}}P_N+\frac{p'(N-1)}{2pN}-\frac{p'}{p}+1} \\
&\times \frac{p'}{p}\frac{\prod_{k=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{N,k}+\frac{p'}{p}-1)}{\prod_{k=1}^{N-1}\Gamma(\sqrt{\frac{p'}{p}}P_{N,k})}\frac{\Gamma(1-\frac{p'}{p})^{N-1}}{\Gamma(\frac{p'}{p}-1)}\, {}_N F_{N-1}(\vec{\mu}_N-\vec{1}-\vec{\delta}_N;\hat{\vec{\nu}}_N-\vec{1}|\frac{x_1}{x_2}).
\end{aligned} \tag{3.223}$$

Using the identity (3.208), $\mathbf{G}_u^{(1),2}$ and $\mathbf{G}_u^{(1),3}$ can be combined into

$$\begin{aligned}
&\mathbf{G}_u^{(1),2} + \mathbf{G}_u^{(1),3} \\
&= \left(\frac{i}{4\pi}\right)^{\frac{p'}{pN}-\frac{p'}{p}N} (x_2-x_1)^{\frac{p'}{Np}} x_1^{\frac{p'(N-1)}{2pN}} x_2^{\frac{p'(N-1)}{2pN}-\frac{p'}{p}} \frac{p'}{p}(1-\frac{p'}{p})(2-\frac{p'}{p}) \sum_{k=1}^{N} \frac{\prod_{i=1}^{N-1}\Gamma(\sqrt{\frac{p'}{p}}P_{N,i}+1)}{\prod_{i=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{N,i}+\frac{p'}{p})} \\
&\times \sum_{m=1}^{N} \frac{\Gamma(\sqrt{\frac{p'}{p}}P_{N,m}+\frac{p'}{p})\Gamma(\sqrt{\frac{p'}{p}}P_{m,N}+1-\frac{p'}{p})\prod_{j=1,j\neq m}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{m,j}+\delta_{m,k}-\delta_{j,k})}{\prod_{j=1}^{N}\Gamma(\sqrt{\frac{p'}{p}}P_{m,j}+1-\frac{p'}{p}+\delta_{j,k}+\delta_{m,k})} \\
&\times e^{i\pi\left(\sqrt{\frac{p'}{p}}P_{m,l}-\frac{p'}{p}\right)} \left(\frac{x_1}{x_2}\right)^{\sqrt{\frac{p'}{p}}P_m-\frac{p'}{p}+\delta_{m,k}} {}_N F_{N-1}(\vec{\mu}'_m-\delta_{k,m}\vec{1}-\vec{\delta}_k;\hat{\vec{\nu}}'_m-\delta_{k,m}\vec{1}+(1-\delta_{k,m})\vec{\delta}_k|\frac{x_2}{x_1}).
\end{aligned} \tag{3.224}$$

3.F Thermal two-point function in Virasoro minimal models

In this appendix, we study numerically the torus two-point function of $(\mathbf{f}, 0)$ with $(\bar{\mathbf{f}}, 0)$, and its analytic continuation to Lorentzian signature, in the $N = 2$ case, i.e. Virasoro minimal model. The result was first derived in [44], and is a special case of our formulae for general N.

The formula in terms of summation over t channel conformal blocks in this case is

$$\begin{aligned}\langle\mathcal{O}_{v_1}(z_1,\bar{z}_1)\mathcal{O}_{v_2}(z_2,\bar{z}_2)\rangle_\tau &= \frac{1}{2}\sum_{r=1}^{p-1}\sum_{s=1}^{p'-1}\left[\left|\mathcal{C}_r\mathcal{G}^t_{\frac{p'r-ps}{\sqrt{2pp'}}}(z_1,z_2|\tau)\right|^2+\left|\mathcal{C}_{-r}\mathcal{G}^t_{\frac{-p'r+ps}{\sqrt{2pp'}}}(z_1,z_2|\tau)\right|^2\right]\\ &=\sum_{r=1}^{p-1}\sum_{s=1}^{p'-1}\left|\mathcal{C}_r\mathcal{G}^t_{\frac{p'r-ps}{\sqrt{2pp'}}}(z_1,z_2|\tau)\right|^2.\end{aligned} \tag{3.225}$$

The subscript of the conformal block $\mathcal{G}^t_u$, $u=\frac{p'r-ps}{\sqrt{2pp'}}$, is the charge associated with the (r,s) primary in the t-channel, normalized such that the fundamental weight is $\frac{1}{\sqrt{2}}$. The normalization factor $\mathcal{C}_r$ is given by

$$\mathcal{C}_r=\frac{1}{\Gamma(1-\frac{p'}{p})}\left[-\gamma(\frac{p'}{p})\gamma(2(1-\frac{p'}{p}))\frac{\sin(\pi\frac{p'}{p}(r-1))}{\sin(\pi\frac{p'}{p}r)}\right]^{\frac{1}{2}}\mathcal{N}_r^{-1}. \tag{3.226}$$

We will also write $\mathcal{G}^t_u$ as $\mathcal{G}^t_{(r,s)}$. It is obtained from the free boson correlator by the contour integral

$$\begin{aligned}\mathcal{G}^t_{(r,s)}(z_1,z_2|\tau) &= \int_{L(0,z_1)}dt\,\mathcal{G}^{bos}_{(r,s)}(z_1,z_2,t|\tau),\\ \mathcal{G}^{bos}_{(r,s)}(z_1,z_2,t|\tau) &= G^{bos}_{(r,s)}(z_1,z_2,t|\tau)-G^{bos}_{(r,-s)}(z_1,z_2,t|\tau).\end{aligned} \tag{3.227}$$

G^{bos} is given explicitly by

$$\begin{aligned}G^{bos}_{(r,s)}(z_1,z_2,t|\tau) &= \frac{1}{\eta(\tau)}\left(\frac{\theta_1(z_{12}|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{\frac{p'}{2p}}\left(\frac{\theta_1(z_1-t|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{-\frac{p'}{p}}\left(\frac{\theta_1(z_2-t|\tau)}{\partial_z\theta_1(0|\tau)}\right)^{-\frac{p'}{p}}\\ &\times\sum_{n=-\infty}^{\infty}q^{pp'(\frac{p'r-ps}{2pp'}+n)^2}\exp\left[2\pi i(\frac{p'r-ps}{2p}+p'n)(z_1+z_2-2t)\right].\end{aligned} \tag{3.228}$$

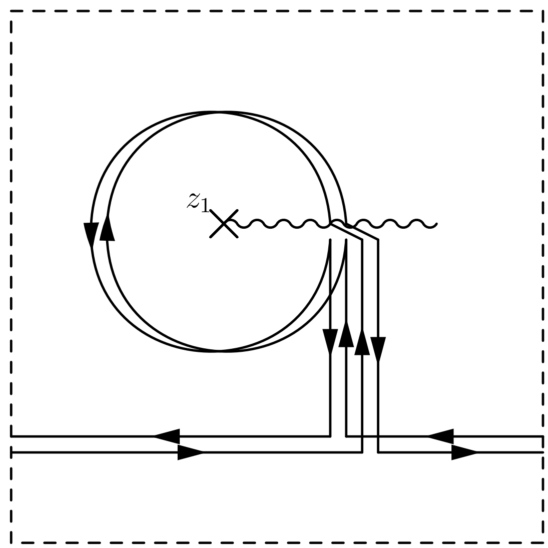
In the explicit evaluation of the two-point function below, we will restrict to the special case $\tau=i\beta$, $z_1=0$, $z_2=1/2$, and compute

$$\mathcal{G}^t_{(r,s)}(0,\frac{1}{2}|i\beta). \tag{3.229}$$

At positive integer values of time, $t = m > 0$, we have

$$\langle \mathcal{O}_{v_1}(0,m)\mathcal{O}_{v_2}(\frac{1}{2},0)\rangle_\beta = \sum_{r=1}^{p-1}\sum_{s=1}^{p'-1} e^{2\pi i m \frac{p'}{p}(-r+\frac{1}{2})} \left| \mathcal{G}^t_{(r,s)}(0,\frac{1}{2}|i\beta) \right|^2. \tag{3.230}$$

The integral is evaluated numerically using the following contour, which is convenient when the fractional powers of $\theta_1(z|\tau)$ in (3.228) is defined with a branch cut along the positive real z axis.



The results for minimal models up to $k = 30$ are plotted in Figures 3.1 and 3.2. At large values of k, while the Poincare recurrence times is of order k, the two-point function is already "thermalized" at $t = 1$.

We also plotted the two-point function at various temperatures, ranging from 0.05 to 20 (times the self-dual temperature), at integer times in the $k = 4$ Virasoro minimal model, in Figure 3.3.

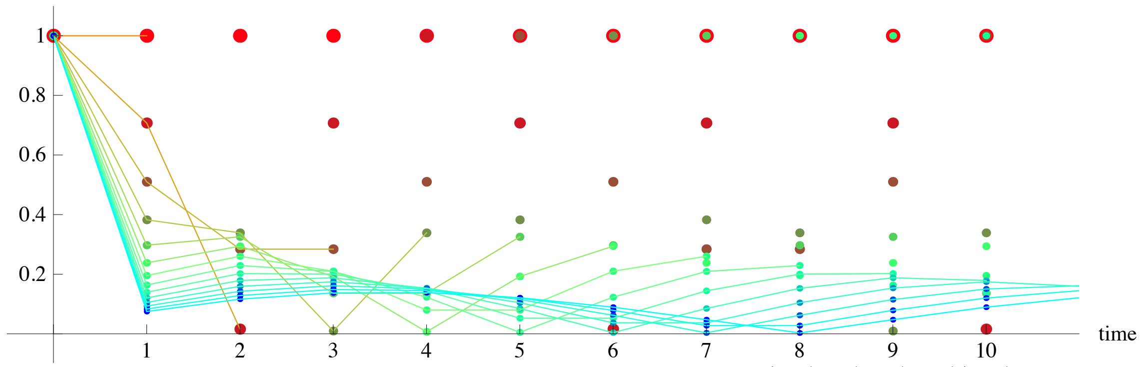


Figure 3.1: The modulus of the two-point function $\langle \mathcal{O}(0,t)\mathcal{O}(0,0)\rangle_\beta$ (normalized to 1 at $t=0$) at inverse temperature $\beta = 0.3$ is plotted at integer values of time $t = 0, 1, 2, \cdots, 10$. The results for Virasoro minimal models with $k = 1, 2, \cdots, 14$ are shown in colors ranging from red to green and then to blue. For each k, the values of the modulus of the two-point function at integer times before Poincaré recurrence are connected with straight lines, for the purpose of illustration only.

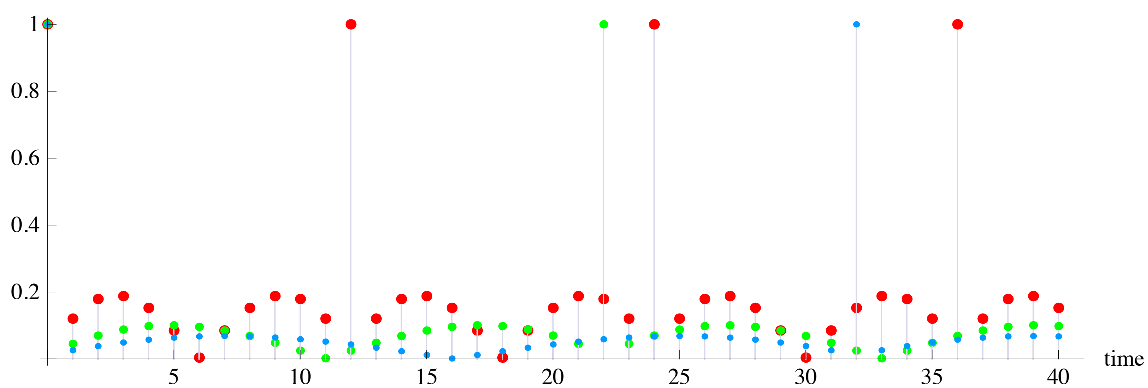


Figure 3.2: The modulus of the two-point function $\langle \mathcal{O}(0,t)\mathcal{O}(0,0)\rangle_\beta$ (normalized to 1 at $t=0$) at inverse temperature $\beta = 0.3$ is plotted at integer values of time $t = 0, 1, 2, \cdots, 40$, in Virasoro minimal models of $k = 10, 20, 30$ (shown in red, green, and blue).

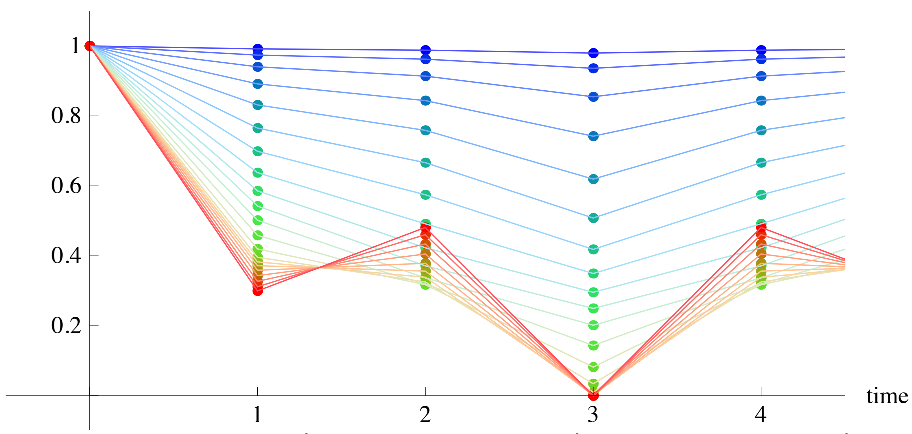


Figure 3.3: Plots of the modulus of the two-point function $\langle \mathcal{O}(0,t)\mathcal{O}(0,0)\rangle_\beta$ (normalized to 1 at $t=0$) in the $k=4$ Virasoro minimal model, at integer values of time $t = 0, 1, \cdots, 4$ (connected with fictitious straight lines for illustration only), at different values of the temperature $T = 1/\beta$. T ranges from ~ 0.05 to 20 (depicted in colors ranging from blue to red), evenly spaced in logarithmic scale.

Chapter 4

A Semi-Local Holographic Minimal Model

4.1 Summary of Section 3.4.3

In previous chapter, we computed the three-point functions of W_N primaries $(\square, 0)$, $(\square, \square)$, and/or their charge conjugates, with the primary (Λ_+, Λ_-) where $\Lambda_\pm$ are $\square\square$ or $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$. This result allowed us to identify the primary operators (Λ_+, Λ_-), for $\Lambda_\pm$ being one- or two-box representations, with the single-particles or multi-particle states in the bulk in large N limit. The result can be summarized in the following table:

$\Lambda_+ \backslash \Lambda_-$	0	$\square$	$\square\square$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$
0	0	$\tilde{\phi}_1$	$L_{\tilde{\phi}_1}$	$\tilde{\phi}_1^2$
$\square$	ϕ_1	ω_1	$\frac{1}{\sqrt{2}}(\tilde{\phi}_1\omega_1 + \tilde{\phi}_2)$	$\frac{1}{\sqrt{2}}(\tilde{\phi}_1\omega_1 - \tilde{\phi}_2)$
$\square\square$	L_{ϕ_1}	$\frac{1}{\sqrt{2}}(\phi_1\omega_1+\phi_2)$	$\frac{1}{2}(\omega_1^2 + \sqrt{2}\omega_2)$	$\frac{1}{\sqrt{2}}(L_{\omega_1} - \frac{1}{\sqrt{2}}(\phi_1\tilde{\phi}_2 - \phi_2\tilde{\phi}_1))$
$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	ϕ_1^2	$\frac{1}{\sqrt{2}}(\phi_1\omega_1-\phi_2)$	$\frac{1}{\sqrt{2}}(L_{\omega_1} + \frac{1}{\sqrt{2}}(\phi_1\tilde{\phi}_2 - \phi_2\tilde{\phi}_1))$	$\frac{1}{2}(\omega_1^2 - \sqrt{2}\omega_2)$

where the $\phi_1, \tilde{\phi}_1, \omega_1, \phi_2, \tilde{\phi}_2, \omega_2$ are operators that dual to the elementary particles in the bulk:

$$\begin{aligned}
&\phi_1 = (\square, 0), \quad \tilde{\phi}_1 = (0, \square), \quad \omega_1 = (\square, \square), \\
&\phi_2 = \frac{1}{\sqrt{2}}\left[(\square\!\square, \square) - (\genfrac{}{}{0pt}{}{\square}{\square}, \square)\right], \quad \tilde{\phi}_2 = \frac{1}{\sqrt{2}}\left[(\square, \square\!\square) - (\square, \genfrac{}{}{0pt}{}{\square}{\square})\right], \\
&\omega_2 = \frac{1}{\sqrt{2}}\left[(\square\!\square, \square\!\square) - (\genfrac{}{}{0pt}{}{\square}{\square}, \genfrac{}{}{0pt}{}{\square}{\square})\right].
\end{aligned} \tag{4.1}$$

Two comments about this identification: first note that the expressions only make sense in the large N limit since each term in the linear combination has different dimension in the subleading order of $1/N$. In the large N limit, we conjecture that each term in the above linear combination has the same dimensions and higher spin charges. This conjecture has been checked up to spin 5; see Appendix 4.A. Second, in the table, the products of the operators are well-defined because one can check that the OPE's of the them have no singularity in the large N limit. The operator $L_{\mathcal{O}}$ is defined as

$$L_{\mathcal{O}} = \frac{1}{2\sqrt{2}h_{\mathcal{O}}}\left(\mathcal{O}\partial\bar{\partial}\mathcal{O} - \partial\mathcal{O}\bar{\partial}\mathcal{O}\right). \tag{4.2}$$

Again, the products are well-defined since there is no singularity in the OPE. This table is further subject to a relation [12]:

$$\frac{1}{2h_{\omega_1}}\partial\bar{\partial}\omega_1 = \phi_1\tilde{\phi}_1. \tag{4.3}$$

The bulk physical meaning of this relation will be explain in detail in the Section 4.5.

In Section 4.2 and 4.3, we will present some new examples of single-trace operators and operator relations involving light primaries at large N. In Section 4.4, we argue that the operator relations that seemed to be in conflict with large N factorization should in fact be interpreted as current non-conservation relations for currents that generate approximate "hidden" symmetries in the large N limit. Further data on higher spin currents of this sort are presented in Section 4.5. In Section 4.6, we state our conjecture on the complete spectrum of single-trace operators in the CFT at infinite N, or single-particle states in the bulk. These include the infinite family of massive scalars $\phi_n, \tilde{\phi}_n$, light scalars ω_n, and the hidden higher spin currents $j_n^{(s)}$, all of which are complex. Various checks based on partition functions and characters are given by Section 4.7. In Section 4.8, we determine the gauge generators associated with the hidden symmetry currents, and reveal the picture of semi-local higher spin theory on $\mathrm{AdS}_3 \times \mathrm{S}^1$. We discuss the implication of our results in Section 4.9.

4.2 New single-trace operators/elementary particles

Let us extend this table to the the representation with three boxes. Before diving into the computation of three-point functions, there are some principles can help us to determine whether a primary operator $\mathcal{O}_A$ can be dual to the two-particle state of two elementary particles that are dual to $\mathcal{O}_B$ and $\mathcal{O}_C$. First, the primary $\mathcal{O}_A$ must appear in the OPE of the primary $\mathcal{O}_B$ and $\mathcal{O}_C$. Second, the dimension of the primary $\mathcal{O}_A$ must be equal to the sum of the dimension of $\mathcal{O}_B$ and $\mathcal{O}_C$ up to higher order corrections in $1/N$. Following is a table summarizing the dimension of the primary operator up to representation of three boxes.

$\Lambda_+ \backslash \Lambda_-$	0	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{1,1}$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
0	0	$\frac{1-\lambda}{2}$	$(1-\lambda)+1$	$1-\lambda$	$3\left(\frac{1-\lambda}{2}\right)+3$	$3\left(\frac{1-\lambda}{2}\right)+1$	$3\left(\frac{1-\lambda}{2}\right)$
$\ydiagram{1}$	$\frac{1+\lambda}{2}$	$\frac{\lambda^2}{2N}$	$\frac{1-\lambda}{2}$	$\frac{1-\lambda}{2}$	$(1-\lambda)+1$	$1-\lambda$	$1-\lambda$
$\ydiagram{2}$	$(1+\lambda)+1$	$\frac{1+\lambda}{2}$	$\frac{\lambda^2}{N}$	1	$\frac{1-\lambda}{2}$	$\frac{1-\lambda}{2}$	$\frac{1-\lambda}{2}+1$
$\ydiagram{1,1}$	$1+\lambda$	$\frac{1+\lambda}{2}$	1	$\frac{\lambda^2}{N}$	$\frac{1-\lambda}{2}+2$	$\frac{1-\lambda}{2}$	$\frac{1-\lambda}{2}$
$\ydiagram{3}$	$3\left(\frac{1+\lambda}{2}\right)+3$	$(1+\lambda)+1$	$\frac{1+\lambda}{2}$	$\frac{1+\lambda}{2}+2$	$\frac{3\lambda^2}{2N}$	1	3
$\ydiagram{2,1}$	$3\left(\frac{1+\lambda}{2}\right)+1$	$1+\lambda$	$\frac{1+\lambda}{2}$	$\frac{1+\lambda}{2}$	1	$\frac{3\lambda^2}{2N}$	1
$\ydiagram{1,1,1}$	$3\left(\frac{1+\lambda}{2}\right)$	$1+\lambda$	$\frac{1+\lambda}{2}+1$	$\frac{1+\lambda}{2}$	3	1	$\frac{3\lambda^2}{2N}$

Let us first focus on the light states: $(\ydiagram{3},\ydiagram{3}), (\ydiagram{2,1},\ydiagram{2,1}), (\ydiagram{1,1,1},\ydiagram{1,1,1})$. By the fusion rule and the additivity of the dimension, two linear combinations of these three operators can be identified with the multi-particle states ω_1^3 and $\omega_1\omega_2$. Let us see this explicitly in terms of structure constants. A formula of a large class of the structure constants is given in [50]. By explicitly evaluating the formula, we find out that, in the large N limit, the OPE of $(\ydiagram{1},\ydiagram{1})$ and $(\ydiagram{2},\ydiagram{2})$ has no singularity, hence the product $(\ydiagram{1},\ydiagram{1})(\ydiagram{2},\ydiagram{2})$ is well-defined, which in the large N limit is

$$(\ydiagram{1},\ydiagram{1})(\ydiagram{2},\ydiagram{2}) = (\ydiagram{3},\ydiagram{3}) + (\ydiagram{2,1},\ydiagram{2,1}). \tag{4.4}$$

Similarly, in the large N limit, we have

$$(\ydiagram{1},\ydiagram{1})(\ydiagram{1,1},\ydiagram{1,1}) = (\ydiagram{2,1},\ydiagram{2,1}) + (\ydiagram{1,1,1},\ydiagram{1,1,1}). \tag{4.5}$$

Rewriting the equation in terms of ω_1, ω_2, we have

$$\begin{aligned} \omega_1\omega_2 &= (\ydiagram{3},\ydiagram{3}) - (\ydiagram{1,1,1},\ydiagram{1,1,1}), \\ \omega_1^3 &= (\ydiagram{3},\ydiagram{3}) + 2(\ydiagram{2,1},\ydiagram{2,1}) + (\ydiagram{1,1,1},\ydiagram{1,1,1}). \end{aligned} \tag{4.6}$$

There is one linear combination of $(\ydiagram{3},\ydiagram{3}), (\ydiagram{2,1},\ydiagram{2,1}), (\ydiagram{1,1,1},\ydiagram{1,1,1})$, which cannot be expressed as $\omega_1\omega_2, \omega_1^3$. This operator should be dual to a new light elementary particle. Hence, we define

$$\omega_3 = \frac{1}{\sqrt{3}}\left[(\ydiagram{3},\ydiagram{3}) - (\ydiagram{2,1},\ydiagram{2,1}) + (\ydiagram{1,1,1},\ydiagram{1,1,1})\right], \tag{4.7}$$

which is orthonormal to $\omega_1\omega_2, \omega_1^3$ and is a new elementary light particle.

Next, let us look at the primaries with dimension $\frac{1-\lambda}{2}$ and with three boxes representations. They are $(\ydiagram{2}, \ydiagram{3}), (\ydiagram{2}, \ydiagram{2,1}), (\ydiagram{1,1}, \ydiagram{2,1}), (\ydiagram{1,1}, \ydiagram{1,1,1})$. From the additivity of the dimension, three linear combinations of these four operators can be dual to the multi-particle states $\tilde{\phi}_1\omega_2, \tilde{\phi}_1\omega_1^2, \tilde{\phi}_2\omega_1$. Again, we can see this explicitly from the structure constants. From the structure constant computation, we have the following products at large N:

$$\begin{aligned}
(0,\ydiagram{1})(\ydiagram{2},\ydiagram{2}) &= \frac{1}{\sqrt{3}}(\ydiagram{2},\ydiagram{3}) + \sqrt{\frac{2}{3}}(\ydiagram{2},\ydiagram{2,1}),\\
(0,\ydiagram{1})(\ydiagram{1,1},\ydiagram{1,1}) &= \sqrt{\frac{2}{3}}(\ydiagram{1,1},\ydiagram{2,1}) + \frac{1}{\sqrt{3}}(\ydiagram{1,1},\ydiagram{1,1,1}),\\
(\ydiagram{1},\ydiagram{1})(\ydiagram{1},\ydiagram{2}) &= \sqrt{\frac{2}{3}}(\ydiagram{2},\ydiagram{3}) + \frac{1}{2\sqrt{3}}(\ydiagram{2},\ydiagram{2,1}) + \frac{\sqrt{3}}{2}(\ydiagram{1,1},\ydiagram{2,1}),\\
(\ydiagram{1},\ydiagram{1})(\ydiagram{1},\ydiagram{1,1}) &= \frac{\sqrt{3}}{2}(\ydiagram{2},\ydiagram{2,1}) + \frac{1}{2\sqrt{3}}(\ydiagram{1,1},\ydiagram{2,1}) + \sqrt{\frac{2}{3}}(\ydiagram{1,1},\ydiagram{1,1,1}).
\end{aligned} \tag{4.8}$$

Expressing them in terms of $\tilde{\phi}_1, \tilde{\phi}_2, \omega_1, \omega_2$, we obtain

$$\begin{aligned}
\tilde{\phi}_1\omega_2 &= \frac{1}{\sqrt{6}}\left[(\ydiagram{2},\ydiagram{3}) + \sqrt{2}(\ydiagram{2},\ydiagram{2,1}) - \sqrt{2}(\ydiagram{1,1},\ydiagram{2,1}) - (\ydiagram{1,1},\ydiagram{1,1,1})\right],\\
\frac{1}{\sqrt{2}}\tilde{\phi}_1\omega_1^2 &= \tilde{\phi}_1\frac{(\ydiagram{2},\ydiagram{2}) + (\ydiagram{1,1},\ydiagram{1,1})}{\sqrt{2}} = \frac{1}{\sqrt{6}}\left[(\ydiagram{2},\ydiagram{3}) + \sqrt{2}(\ydiagram{2},\ydiagram{2,1}) + \sqrt{2}(\ydiagram{1,1},\ydiagram{2,1}) + (\ydiagram{1,1},\ydiagram{1,1,1})\right]\\
&= \frac{1}{\sqrt{2}}\omega_1\frac{[(\ydiagram{1},\ydiagram{2}) + (\ydiagram{1},\ydiagram{1,1})]}{\sqrt{2}} = \frac{1}{\sqrt{6}}\left[(\ydiagram{2},\ydiagram{3}) + \sqrt{2}(\ydiagram{2},\ydiagram{2,1}) + \sqrt{2}(\ydiagram{1,1},\ydiagram{2,1}) + (\ydiagram{1,1},\ydiagram{1,1,1})\right],\\
\tilde{\phi}_2\omega_1 &= \frac{1}{\sqrt{6}}\left[\sqrt{2}(\ydiagram{2},\ydiagram{3}) - (\ydiagram{2},\ydiagram{2,1}) + (\ydiagram{1,1},\ydiagram{2,1}) - \sqrt{2}(\ydiagram{1,1},\ydiagram{1,1,1})\right].
\end{aligned} \tag{4.9}$$

There is one linear combination of $(\ydiagram{2}, \ydiagram{3}), (\ydiagram{2}, \ydiagram{2,1}), (\ydiagram{1,1}, \ydiagram{2,1}), (\ydiagram{1,1}, \ydiagram{1,1,1})$, which is linear independent of $\tilde{\phi}_1\omega_2, \tilde{\phi}_1\omega_1^2, \tilde{\phi}_2\omega_1$, and should be dual to a new elementary particle in the bulk. Hence, we can define

$$\tilde{\phi}_3 = \frac{1}{\sqrt{6}}\left[\sqrt{2}(\ydiagram{2},\ydiagram{3}) - (\ydiagram{2},\ydiagram{2,1}) - (\ydiagram{1,1},\ydiagram{2,1}) + \sqrt{2}(\ydiagram{1,1},\ydiagram{1,1,1})\right], \tag{4.10}$$

which is orthonormal to $\tilde{\phi}_1\omega_2, \frac{1}{\sqrt{2}}\tilde{\phi}_1\omega_1^2, \tilde{\phi}_2\omega_1$. Similarly, by exchanging the left and right representations, we have

$$\begin{aligned}
\phi_1\omega_2 &= \frac{1}{\sqrt{6}}\left[(\ydiagram{3},\ydiagram{2}) + \sqrt{2}(\ydiagram{2,1},\ydiagram{2}) - \sqrt{2}(\ydiagram{2,1},\ydiagram{1,1}) - (\ydiagram{1,1,1},\ydiagram{1,1})\right], \\
\frac{1}{\sqrt{2}}\phi_1\omega_1^2 &= \frac{1}{\sqrt{6}}\left[(\ydiagram{3},\ydiagram{2}) + \sqrt{2}(\ydiagram{2,1},\ydiagram{2}) + \sqrt{2}(\ydiagram{2,1},\ydiagram{1,1}) + (\ydiagram{1,1,1},\ydiagram{1,1})\right], \\
\phi_2\omega_1 &= \frac{1}{\sqrt{6}}\left[\sqrt{2}(\ydiagram{3},\ydiagram{2}) - (\ydiagram{2,1},\ydiagram{2}) + (\ydiagram{2,1},\ydiagram{1,1}) - \sqrt{2}(\ydiagram{1,1,1},\ydiagram{1,1})\right],
\end{aligned} \tag{4.11}$$

and we define

$$\phi_3 = \frac{1}{\sqrt{6}}\left[\sqrt{2}(\ydiagram{3},\ydiagram{2}) - (\ydiagram{2,1},\ydiagram{2}) - (\ydiagram{2,1},\ydiagram{1,1}) + \sqrt{2}(\ydiagram{1,1,1},\ydiagram{1,1})\right]. \tag{4.12}$$

Next, let us focus on the primaries $(\ydiagram{1},\ydiagram{2,1}), (\ydiagram{1},\ydiagram{1,1,1})$. By the fusion rule and the additivity of the dimension, it is not hard to see that they must be identified with the two linear combinations of $\tilde{\phi}_1\tilde{\phi}_2$ and $\omega_1\tilde{\phi}_1^2$, which are dual to two- and three-particle states. Similarly, the primaries $(\ydiagram{2,1},\ydiagram{1}), (\ydiagram{1,1,1},\ydiagram{1})$ are identified with the two linear combinations of $\phi_1\phi_2$ and $\omega_1\phi_1^2$. All the other primaries: $(\ydiagram{1},\ydiagram{3})$, $(\ydiagram{2},\ydiagram{1,1,1})$, $(\ydiagram{1,1},\ydiagram{3})$, $(\ydiagram{3},\ydiagram{2,1})$, $(\ydiagram{3},\ydiagram{1,1,1})$, $(\ydiagram{2,1},\ydiagram{1,1,1})$, and the primaries with left and right representations exchanged, are also dual to multi-particle states. We will show this in Section 4.7.

4.3 Large N operator relations involving ω_2 and ω_3

There is a new relation involving the descendant of ω_2, similar to the relation (4.3). By the following two structure constants:

$$\begin{aligned}
C_{nor}\Big((0,\ydiagram{1}),(\ydiagram{1,1},\ydiagram{1}),(\overline{\ydiagram{1,1}},\overline{\ydiagram{1,1}})\Big) &= \frac{\sqrt{2}}{N} + \mathcal{O}(\frac{1}{N^2}), \\
C_{nor}\Big((0,\ydiagram{1}),(\ydiagram{2},\ydiagram{1}),(\overline{\ydiagram{2}},\overline{\ydiagram{2}})\Big) &= \frac{\sqrt{2}}{N} + \mathcal{O}(\frac{1}{N^2}),
\end{aligned} \tag{4.13}$$

we have the three-point functions:

$$\left\langle \bar{\omega}_2(z)\phi_1(w)\tilde{\phi}_2(0) \right\rangle = \left\langle \bar{\omega}_2(z)\phi_2(w)\tilde{\phi}_1(0) \right\rangle = \frac{1}{\sqrt{2}N}\frac{1}{|z-w|^{2\lambda}|w|^2|z|^{-2\lambda}}, \tag{4.14}$$

in the large N limit. Taking $\partial\bar{\partial}$ on $\bar{\omega}_2$, we obtain:

$$\left\langle \partial\bar{\partial}\bar{\omega}_2(z)\phi_1(w)\tilde{\phi}_2(0) \right\rangle = \left\langle \partial\bar{\partial}\bar{\omega}_2(z)\phi_2(w)\tilde{\phi}_1(0) \right\rangle = \frac{\lambda^2}{\sqrt{2}N}\left(\frac{1}{|z-w|^{2(1+\lambda)}}\right)\left(\frac{1}{|z|^{2(1-\lambda)}}\right). \tag{4.15}$$

The two factors on the right hand side of (4.15) are precisely given by the two-point functions of $\langle \phi_2\bar{\phi}_2\rangle$ and $\left\langle \tilde{\phi}_1\bar{\tilde{\phi}}_1\right\rangle$, or $\langle \phi_1\bar{\phi}_1\rangle$ and $\left\langle \tilde{\phi}_2\bar{\tilde{\phi}}_2\right\rangle$. Hence, this suggests the following relation in the large N limit:

$$\frac{1}{2h_{\omega_2}}\partial\bar{\partial}\omega_2 = \frac{1}{\sqrt{2}}(\phi_1\tilde{\phi}_2 + \tilde{\phi}_1\phi_2). \tag{4.16}$$

To make sure that there are no extra term on the left hand side, one can compute the two-point function for the right hand side of (4.16) with its charge conjugate, and the two-point function for the left hand side of (4.16) with its charge conjugate, and find that they agree.

Form the previous analysis on ω_1, ω_2, it suggests that there is also a relation involving the descendant of ω_3. We postulate such relation should be

$$\frac{1}{2h_{\omega_3}}\partial\bar{\partial}\omega_3 = \frac{1}{\sqrt{3}}(\phi_1\tilde{\phi}_3 + \phi_2\tilde{\phi}_2 + \phi_3\tilde{\phi}_1). \tag{4.17}$$

We give an argument for this relation. In the large N limit, we have the following structure constants

$$\begin{aligned} &C_{nor}\Big((0,\ydiagram{1}),(\ydiagram{3},\ydiagram{2}),(\overline{\ydiagram{3}},\overline{\ydiagram{3}})\Big) = \frac{\sqrt{3}}{N}, \quad C_{nor}\Big((0,\ydiagram{1}),(\ydiagram{1,1,1},\ydiagram{1,1}),(\overline{\ydiagram{1,1,1}},\overline{\ydiagram{1,1,1}})\Big) = \frac{\sqrt{3}}{N} \\ &C_{nor}\Big((0,\ydiagram{1}),(\ydiagram{2,1},\ydiagram{2}),(\overline{\ydiagram{2,1}},\overline{\ydiagram{2,1}})\Big) = \sqrt{\frac{3}{2}}\frac{1}{N}, \quad C_{nor}\Big((0,\ydiagram{1}),(\ydiagram{2,1},\ydiagram{1,1}),(\overline{\ydiagram{2,1}},\overline{\ydiagram{2,1}})\Big) = \sqrt{\frac{3}{2}}\frac{1}{N}. \end{aligned} \tag{4.18}$$

These structure constants give the three-point functions:

$$\left\langle \bar{\omega}_3(z)\phi_1(w)\tilde{\phi}_3(0) \right\rangle = \left\langle \bar{\omega}_3(z)\phi_3(w)\tilde{\phi}_1(0) \right\rangle = \frac{1}{\sqrt{3}N}\frac{1}{|z-w|^{2\lambda}|w|^2|z|^{-2\lambda}}, \tag{4.19}$$

in the large N limit. Taking $\partial\bar{\partial}$ on $\bar{\omega}_2$, the three-point function again factorizes as a product of two two-point functions:

$$\left\langle \partial\bar{\partial}\bar{\omega}_3(z)\phi_1(w)\tilde{\phi}_3(0)\right\rangle = \left\langle \partial\bar{\partial}\bar{\omega}_3(z)\phi_3(w)\tilde{\phi}_1(0)\right\rangle = \frac{\lambda^2}{\sqrt{3}N}\frac{1}{|z-w|^{2(1+\lambda)}|z|^{2(1-\lambda)}}. \tag{4.20}$$

The three-point functions (4.20) imply the relation

$$\frac{1}{2h_{\omega_3}}\partial\bar{\partial}\omega_3 = \frac{1}{\sqrt{3}}(\phi_1\tilde{\phi}_3 + \phi_3\tilde{\phi}_1 + \cdots). \tag{4.21}$$

By comparing the two-point functions of the left and right hand sides with their charge conjugates, we know that the "$\cdots$" must take the form as a single term $\phi_n\tilde{\phi}_m$ with $n, m \neq 1, 3$, and the only candidate is $\phi_2\tilde{\phi}_2$.

4.4 Hidden symmetries

In this section, we give physical interpretation of the relations (4.3), (4.16), (4.17), and provide a bulk mechanism of producing such relations. The key observation is that the dimension of ω_n goes to zero in the large N limit. Therefore, it should effectively behave like a free boson, whose derivative is a conversed current. Hence, we define the holomorphic current $(j_n^{(1)})_z = \partial\omega_n/\sqrt{2h_{\omega_n}}$ and also the antiholomorphic current $(j_n^{(1)})_{\bar{z}} = \bar{\partial}\omega_n/\sqrt{2h_{\omega_n}}$, for $n = 1, 2, 3$, which has normalized two-point function with itself. For simplicity, we will sometimes suppress the index by simply denoting $(j_n^{(1)})_z$ as $j_n^{(1)}$ in the following. However, since the dimensions of ω_n are not exactly equal to zero, the currents $j_n^{(1)}$ are not exactly conserved. The relations (4.3), (4.16), (4.17) are then naturally interpreted as current non-conservation equations[1]:

$$\bar{\partial} j_n^{(1)} = \frac{\lambda}{\sqrt{N}}(\phi_1 \tilde{\phi}_n + \phi_2 \tilde{\phi}_{n-1} + \cdots + \phi_n \tilde{\phi}_1). \tag{4.22}$$

The bulk interpretation of these current non-conservation equations is simple. Let us illustrate this by considering the case of $j_n^{(1)}$. In this case the current non-conservation equation is simply

$$\bar{\partial} j_1^{(1)} = \frac{\lambda}{\sqrt{N}} \phi_1 \tilde{\phi}_1. \tag{4.23}$$

Following the AdS/CFT dictionary, the bulk dual of the current $j_1^{(1)}$ is a $U(1)$ Chern-Simons gauge field A_μ, and the bulk dual of the operators $\phi_1, \tilde{\phi}_1$ are two scalars $\Phi, \widetilde{\Phi}$. These two scalars have different but complementary dimensions, hence they have the same mass but different boundary conditions. They can be minimally coupled to the gauge field A_μ. The action of this system up to cubic order is

$$S = \frac{k_{CS}}{4\pi} \int AdA + 2i \int d^2x dz \sqrt{g} A^\mu \left[\widetilde{\Phi} \partial_\mu \Phi - \Phi \partial_\mu \widetilde{\Phi} \right]. \tag{4.24}$$

Using this action, we can compute the three-point function of $\bar{\partial} j_1^{(1)}$ with $\phi_1, \tilde{\phi}_1$. The boundary to bulk propagator of the Chern-Simons gauge field takes a pure gauge form $A_\mu = \partial_\mu \Lambda$. The cubic action, hence, can be written as

$$\lim_{z \to 0} \frac{2}{z} \int d^2x \Lambda \left[\Phi \partial_z \widetilde{\Phi} - \widetilde{\Phi} \partial_z \Phi \right]. \tag{4.25}$$

[1] The current non-conservations equation for theories in one higher dimension have been studied in [21, 51, 52].

The three-point function is then given by

$$\begin{aligned}
&\left\langle j_1^{(1)}(\vec{x}_3)\phi_1(x_1)\tilde{\phi}_1(x_2)\right\rangle \\
&= \lim_{z\to 0}\frac{2}{z}\int d^2x\Lambda(x-x_3)\left[K_{1+\lambda}(x-x_1)\partial_z K_{1-\lambda}(x-x_2)-K_{1-\lambda}(x-x_2)\partial_z K_{1+\lambda}(x-x_1)\right] \\
&= -16\pi\lambda\int d^2x\frac{1}{(x^+-x_3^+)}\frac{1}{|\vec{x}-\vec{x}_2|^{2(1-\lambda)}}\frac{1}{|\vec{x}-\vec{x}_1|^{2(1+\lambda)}},
\end{aligned} \tag{4.26}$$

where K_Δ and Λ are the boundary to bulk propagators for the scalar and gauge function:

$$K_\Delta = \left(\frac{z}{z^2+|\vec{x}|^2}\right)^\Delta, \quad \Lambda = \frac{4\pi}{x^+}. \tag{4.27}$$

Taking the derivative $\frac{\partial}{\partial x_3^+}$ on the above expression, we obtain

$$\begin{aligned}
\left\langle \bar{\partial} j_1^{(1)}(\vec{x}_3)\phi_1(x_1)\tilde{\phi}_1(x_2)\right\rangle &= -16\pi^2\lambda\int d^2x\delta^2(x-x_3)\frac{1}{|\vec{x}-\vec{x}_2|^{2(1-\lambda)}}\frac{1}{|\vec{x}-\vec{x}_1|^{2(1+\lambda)}} \\
&= -16\pi^2\lambda\frac{1}{|\vec{x}_3-\vec{x}_2|^{2(1-\lambda)}|\vec{x}_3-\vec{x}_1|^{2(1+\lambda)}},
\end{aligned} \tag{4.28}$$

which factories into a product of two two-point functions of scalars with dimension $\Delta = 1+\lambda$ and $1-\lambda$. This matches exactly with what we expected from (4.23) provided the identification of the Chern-Simons level $k_{CS} = N$. In Section 4.8, we will show that every $(j_n^{(1)})_z$ gives a $U(1)$ Chern-Simons gauge field, and combined with the gauge field dual to $(j_n^{(1)})_{\bar{z}}$, they form a $U(1)^\infty \times U(1)^\infty$ Chern-Simons gauge theory in the bulk.

4.5 Approximately conserved higher spin currents

The approximately conserved spin-1 current $(j_n^{(1)})_z$ generates a tower of approximately conserved higher spin currents $(j_n^{(s)})_z$, by the action of W_N generators on $(j_n^{(1)})_z$. For example,

$(j_1^{(1)})_z$ has a level-one W-descendent

$$\begin{aligned}(j_1^{(2)})_z &= \frac{1}{\sqrt{2(1-\lambda^2)}}\left(W_{-1}^{(3)} - \frac{3}{2}i\lambda L_{-1}\right)(j_1^{(1)})_z \\ &= \sqrt{\frac{N}{2\lambda^2(1-\lambda^2)}}(W_{-2}^{(3)} - i\lambda\partial^2)\omega_1,\end{aligned} \tag{4.29}$$

which is also a Virasoro primary[2]. This is an approximately conserved stress tensor. The current non-conservation equation of $(j_1^{(1)})_z$ then descends to the current non-conservation equation of $(j_1^{(2)})_z$:

$$\begin{aligned}\bar\partial(j_1^{(2)})_z &= \frac{1}{\sqrt{2(1-\lambda^2)}}\left(W_{-1}^{(3)} - \frac{3}{2}i\lambda L_{-1}\right)\bar\partial j_1^{(1)} \\ &= \frac{i\lambda}{\sqrt{2N(1-\lambda^2)}}\left[(1-\lambda)\partial\phi_1\tilde\phi_1 - (1+\lambda)\phi_1\partial\tilde\phi_1\right],\end{aligned} \tag{4.30}$$

where we have used the null-state equations in Appendix 4.C. In general, the approximately conserved spin-1 current $(j_1^{(1)})_z$ has exactly one W-descendant Virasoro primary $(j_1^{(s)})_z$ at each level s, which takes the form as

$$(j_1^{(s)})_z = \sqrt{N}(a_1 W_{-s}^{(s+1)} + a_2\partial W_{-s+1}^{(s)} + \cdots + a_s\partial^s)\omega_1, \tag{4.31}$$

where a_i are some constants depending on λ, and can be fixed by requiring $(j_1^{(s)})_z$ being a Virasoro primary. The $(j_1^{(s)})_z$'s are approximately conserved higher spin currents. They satisfy the current non-conservation equations taking the form as

$$\bar\partial(j_1^{(s)})_z = \frac{1}{\sqrt{N}}(b_1\partial^{s-1}\phi_1\tilde\phi_1 + b_2\partial^{s-2}\phi_1\partial\tilde\phi_1 + \cdots + b_s\phi_1\partial^{s-1}\tilde\phi_1), \tag{4.32}$$

where b_s are constants depending on λ, and can be fixed by requiring the left hand side of (4.32) being a Virasoro primary. By same argument, there are also antiholomorphic higher spin currents $(j_1^{(s)})_{\bar z}$. We expect that there are also approximately conserved holomorphic

[2]In Appendix 4.B, we fix the normalization of $(j_1^{(1)})_z$ and check that it is a Virasoro primary.

and antiholomorphic higher spin currents $(j_2^{(s)})_z$, $(j_3^{(s)})_z$, and $(j_2^{(s)})_{\bar z}$, $(j_3^{(s)})_{\bar z}$ that take a the similar form as (4.31).

4.6 The single particle spectrum

Now we state a conjecture on the complete spectrum of the single particle states in the bulk. Throughout this paper, by a single-trace operator we mean an operator that obeys the same large N factorization property as single-trace operators in large N gauge theories; such an operator is dual to the state of one elementary particle in the bulk. The products of single-trace operators are dual to multi-particle states. As we have seen in the previous section, the primary operators that involve up to one box in the Young tableaux of Λ_+ and Λ_- are all single-trace operators: they are ϕ_1, $\tilde\phi_1$, and ω_1. The primaries that involve up to two boxes in the Young tableaux of Λ_+ and Λ_- are some suitable linear combination of single-trace operators ϕ_2, $\tilde\phi_2$, ω_2, or products of two single-trace operators. We have also seen some evidences that the primaries with up to three boxes in their representations are linear combinations of single-trace operators ϕ_3, $\tilde\phi_3$, ω_3, or products of single-trace operators. We conjecture that the primaries with up to n-box representations are linear combinations of single-trace operators ϕ_n, $\tilde\phi_n$, ω_n, or products of such single-trace operators $\phi_m, \tilde\phi_m, \omega_m$ for $m < n$. Here ϕ_n is a linear combination of primaries of the form (Λ_+, Λ_-) that involve $(n, n-1)$ boxes, $\tilde\phi_n$ is a linear combination of primaries that involve $(n-1, n)$ boxes, and ω_n is a linear combination of light primaries of the form (Λ, Λ) where Λ involves n boxes.

A part of this conjecture is easy to prove: the statement that there is only one light single-trace operator ω_n for each n labeling the number of boxes in its corresponding $SU(N)$ representations follows easily from the fusion rule. First we note that generally, the light state of the form (Λ, Λ) have dimension $B(\Lambda)\lambda^2/N + \mathcal{O}(N^{-2})$, where $B(\Lambda)$ is the number of boxes of the Young tableaux of the representation Λ, in the large N limit and fixed finite $B(\Lambda)$. We may write a partition function of the light states

$$Z(x) = \sum_{(\Lambda,\Lambda)} x^{B(\Lambda)} = \prod_{n=1}^{\infty} \frac{1}{1-x^n}. \tag{4.33}$$

Each single-trace operator of dimension $n\lambda^2/N$ is a linear combination of (Λ, Λ) with $B(\Lambda) = n$. The dimension of the product of single-trace operators is additive at order $1/N$. The products of a single-trace operator is counted by the partition function $1/(1-x^n)$. By comparing this with $Z(x)$, we see that there is precisely one single-trace operator ω_n for each n.

The ϕ_n, $\tilde{\phi}_n$, ω_n are all the single-trace operators that are dual to scalar fields in the bulk. These are not all, however. There are other single-trace operators that are dual to spin-1, spin-2, and higher spin gauge fields. As explained in the previous section, while $\partial\omega_n$ is a level-one descendent of ω_n, the norm of $\partial\omega_n$ goes to zero in the large N limit. Consequently, the normalized operator $(j_n^{(1)})_z \sim \sqrt{N}\partial\omega_n$ behaves like a primary operator. Such operators will be referred to as *large N primary operators*, and we include them in our list of single-trace operators because they should be dual to elementary fields in the bulk as well. We conjecture that $j_n^{(1)}$'s are single-trace operators dual to the spin-1 Chern-Simons gauge field in take bulk. This statement has passed some tests involving the three-point function of $j_n^{(1)}$ with two scalars. This is not the end of the story. As shown in the previous section, there are large N primaries of higher spin s, denoted by $j_n^{(s)}$. These are single-trace operators dual to additional elementary higher spin gauge fields in the bulk. Unlike the original W_N currents, however, the would-be higher spin symmetries generated by $j_n^{(s)}$ are broken by the boundary conditions on the charged scalars, leading to the current non-conservation relation. These hidden symmetries are recovered in the infinite N limit.

Let us summarize our conjecture on the single-particle spectrum. There are two families of complex single-trace operators $\phi_n, \tilde{\phi}_n$, which are dual to massive complex scalar fields (of the same mass classically), one family of complex single-trace operators ω_n, that are dual to massless scalars in the bulk, and a family of approximately conserved higher spin single-trace operators $j_n^{(s)}$ for each positive integer spin $s = 1, 2, 3, \cdots$, that are dual to Chern-Simons spin-1 and higher spin gauge fields.

4.7 Large N partition functions

In this section, we check our proposed single particle spectrum against the partition function of the W_N minimal model in the large N limit.

Let us consider a single-trace operator $\mathcal{O}$ with nonzero left and right dimensions $h_{\mathcal{O}}$ and $\bar{h}_{\mathcal{O}}$. $\mathcal{O}$ is dual to the ground state of a single elementary particle in the bulk. The $SL(2,\mathbb{C})$ descendent operators $\partial^m \bar{\partial}^n \mathcal{O}$ are dual to the excited states of that elementary particle. The contribution of this single elementary particle to the partition function is given by

$$Z_{\mathcal{O}} = \frac{q^{h_{\mathcal{O}}} \bar{q}^{\bar{h}_{\mathcal{O}}}}{(1-q)(1-\bar{q})}. \tag{4.34}$$

If a single-trace operator j has zero right (or left) conformal dimension, then $\partial^m j$ (or $\bar{\partial}^m j$) are all its $SL(2,\mathbb{C})$ dependents. The contribution of j to the partition function is then given by

$$Z_j = \frac{q^{h_j}}{1-q} \quad (\text{or} \quad \frac{\bar{q}^{\bar{h}_j}}{1-\bar{q}}\ ). \tag{4.35}$$

If a single-trace operator ω has zero left and right conformal dimension, then it has no $SL(2,\mathbb{C})$ dependent. The contribution of ω to the partition function is given by $Z_\omega = 1$.

According to our conjecture, we have the single-trace operators $\{\phi_n, \tilde{\phi}_n, \omega_n, (j_n^{(s)})_z, (j_n^{(s)})_{\bar{z}}\}$. Their contributions to the partition are given by

$$Z_{\phi_n} = \frac{q^{\frac{1+\lambda}{2}} \bar{q}^{\frac{1+\lambda}{2}}}{(1-q)(1-\bar{q})}, \quad Z_{\tilde{\phi}_n} = \frac{q^{\frac{1-\lambda}{2}} \bar{q}^{\frac{1-\lambda}{2}}}{(1-q)(1-\bar{q})}, \tag{4.36}$$

and

$$Z_{\omega_n} = 1, \quad Z_{(j_n^{(s)})_z} = \frac{q^s}{1-q}, \quad Z_{(j_n^{(s)})_{\bar{z}}} = \overline{Z_{(j_n)_z}} = \frac{\bar{q}^s}{1-\bar{q}} \tag{4.37}$$

For simplicity, let us sum up the partition functions of the operators $(j_n^{(s)})_z$ to a single partition function $Z_{(j_n)_z}$ as

$$Z_{(j_n)_z} = \sum_{s=1}^{\infty} Z_{(j_n^{(s)})_z} = \sum_{s=1}^{\infty} \frac{q^s}{1-q} = \frac{q}{(1-q)^2}, \tag{4.38}$$

and similar for operators $(j_n^{(s)})_{\bar{z}}$. The bulk theory also contain boundary higher spin gauge fields. Their contribution to the partition function is given by

$$Z_{hs} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{(1-q^n)(1-\bar{q}^n)}. \tag{4.39}$$

Next, let us consider the partition function for the W_N minimal model in the large N limit. Following from the diagonal modular invariance, the partition function in the large N limit is given by the sum of the absolute value square of the characters:

$$Z_{W_N} = \sum_{(\Lambda_+, \Lambda_-)} |\chi_{(\Lambda_+, \Lambda_-)}|^2. \tag{4.40}$$

The characters $\chi_{(\Lambda_+, \Lambda_-)}$, for $\Lambda_\pm$ being representations with one to three boxes in the Young tableaux, in the large N limit are computed in the Appendix 4.D up to cubic order. The following formulas in this section have all been checked up to this order. Let us start by looking at the contribution of the identity operator to the partition function, which in the large N limit gives the partition function of the boundary higher spin gauge fields:

$$\lim_{N\to\infty} |\chi_{(0,0)}|^2 = Z_{hs}. \tag{4.41}$$

The primary operators $(\square, 0) = \phi_1$ and $(0, \square) = \tilde{\phi}_1$ are dual to massive scalars. Their contributions to the partition function indeed give the partition function of single massive scalar (with boundary higher spin gauge fields)

$$\lim_{N\to\infty} |\chi_{(\square,0)}|^2 = Z_{hs} Z_{\phi_1}, \qquad \lim_{N\to\infty} |\chi_{(0,\square)}|^2 = Z_{hs} Z_{\tilde{\phi}_1}. \tag{4.42}$$

The primary operator $(\square, \square) = \omega_1$ is dual to a massless scalar. The W_N-descendants $j_1^{(s)}$ of $(\square, \square)$ are dual to spin-1, spin-2 and higher spin gauge fields. The other W_N descendants of $(\square, \square)$ are dual to two-particle states, by the equation (4.32). We confirm this by the following decomposition of the character,

$$\lim_{N\to\infty} |\chi_{(\square,\square)}|^2 = Z_{hs}(Z_{\omega_1} + Z_{(j_1)_z} + \overline{Z_{(j_1)_z}} + Z_{\phi_1} Z_{\tilde{\phi}_1}), \tag{4.43}$$

where the last term is the contribution of the two-particle states of ϕ_1 and $\tilde{\phi}_1$.

The identification of other primary operators are inevitable involving multi-particle states. By Bose statistics, we can write a multi-particle partition function in terms of the single-particle partition function (4.34) as

$$Z^{multi}_{\mathcal{O}}(t) = \exp\left[\sum_{m=1}^{\infty} \frac{Z_{\mathcal{O}}(q^m, \bar{q}^m)}{m} t^m\right]. \tag{4.44}$$

Suppose $\mathcal{O} = \phi_n$, then the partition function $Z^{multi}_{\phi_n}(t)$ can be expanded as

$$Z^{multi}_{\phi_n}(t) = \sum_{\ell=0}^{\infty} t^\ell Z_{\phi_n^\ell}, \tag{4.45}$$

where $Z_{\phi_n^m}$ is the m-particle partition function. For instance, $Z_{\phi_n^2}$ and $Z_{\phi_n^3}$ are given by

$$\begin{aligned} Z_{\phi_n^2} &= \frac{q^{1+\lambda}\bar{q}^{1+\lambda}(1+q\bar{q})}{(1-q)^2(1+q)(1-\bar{q})^2(1+\bar{q})}, \\ Z_{\phi_n^3} &= \frac{q^{\frac{3}{2}(1+\lambda)}\bar{q}^{\frac{3}{2}(1+\lambda)}(1+q\bar{q}+q^2\bar{q}+\bar{q}^2 q+q^2\bar{q}^2+q^3\bar{q}^3)}{(1-q)^3(1+q)(1+q+q^2)(1-\bar{q})^3(1+\bar{q})(1+\bar{q}+\bar{q}^2)}. \end{aligned} \tag{4.46}$$

For $\mathcal{O} = \omega_n$, all the m-particle partition functions are identity:

$$Z_{\omega_n^m} = 1. \tag{4.47}$$

For $\mathcal{O} = j_n^{(s)}$, the multi-particle partition function for $j_n^{(s)}$, $s = 1, 2, \cdots$, can be computed from

$$Z_{j_n}^{multi}(t) = \prod_{s=1}^{\infty} Z_{j_n^{(s)}}^{multi}(t) = \exp\left[\sum_{m=1}^{\infty}\sum_{s=1}^{\infty} \frac{\chi_{j_n^{(s)}}^{\infty}(q^m)}{m} t^m\right] = \exp\left[\sum_{m=1}^{\infty} \frac{\chi_{j_n}^{\infty}(q^m)}{m} t^m\right]. \tag{4.48}$$

Expanding $Z_{j_n}^{multi}(t)$ in powers of t, we can write the $Z_{j_n}^{multi}(t)$ as

$$Z_{j_n}^{multi}(t) = 1 + Z_{(j_n)_z} t + Z_{(j_n)_z^2} t^2 + Z_{(j_n)_z^3} t^3 + \cdots, \tag{4.49}$$

where $Z_{(j_n)_z^m}$ has the interpretation of the m-particle partition function. For instance,

$$\begin{aligned} Z_{(j_n)_z^2} &= \frac{q^2(1+q^2)}{(1-q)^4(1+q)^2}, \\ Z_{(j_n)_z^3} &= \frac{q^3(1+q^2+2q^2+q^4+q^6)}{(1-q)^6(1+q)^2(1+q+q^2)^2}. \end{aligned} \tag{4.50}$$

Let us continue on the matching of boundary and bulk partition functions. Consider the primary operators $(\ydiagram{1,1}, 0)$ and $(\ydiagram{2}, 0)$. They are dual to two-particle states. Their contribution to the partition function matches with the two particle partition function:

$$\lim_{N\to\infty}\left(|\chi_{(\ydiagram{2},0)}|^2 + |\chi_{(\ydiagram{1,1},0)}|^2\right) = Z_{hs} Z_{\phi_1^2}. \tag{4.51}$$

Now, consider the primary operators $(\ydiagram{3}, 0)$, $(\ydiagram{2,1}, 0)$, and $(\ydiagram{1,1,1}, 0)$. They are dual to three-particle states. Their contribution to the partition function matches with the three-particle partition function:

$$\lim_{N\to\infty}\left(|\chi_{(\ydiagram{3},0)}|^2 + |\chi_{(\ydiagram{2,1},0)}|^2 + |\chi_{(\ydiagram{1,1,1},0)}|^2\right) = Z_{hs} Z_{\phi_1^3}. \tag{4.52}$$

Next, consider the primary operators $(\yng(2), \yng(1))$ and $(\yng(1,1), \yng(1))$. Their contribution to the partition function also decomposes as the multi-particle partition functions:

$$\lim_{N\to\infty}\left(|\chi_{(\yng(2),\yng(1))}|^2 + |\chi_{(\yng(2),\yng(1))}|^2\right) = Z_{hs}\Big[Z_{\phi_1}\left(Z_{\omega_1} + Z_{(j_1)_z} + \overline{Z_{(j_1)_{\bar{z}}}}\right) + Z_{\tilde{\phi}_1}Z_{\phi_1^2} + Z_{\phi_2}\Big]. \tag{4.53}$$

For the primary operators $(\yng(2), \yng(2))$, $(\yng(1,1), \yng(1,1))$, $(\yng(2), \yng(1,1))$, and $(\yng(1,1), \yng(2))$, their contribution to the partition function decomposes as

$$\begin{aligned}
&\lim_{N\to\infty}\left(|\chi_{(\yng(2),\yng(2))}|^2 + |\chi_{(\yng(1,1),\yng(1,1))}|^2 + |\chi_{(\yng(2),\yng(1,1))}|^2 + |\chi_{(\yng(1,1),\yng(2))}|^2\right)\\
&= Z_{hs}\Big[Z_{\omega_1^2} + Z_{\omega_1}(Z_{(j_1)_z} + \overline{Z_{(j_1)_z}}) + (Z_{(j_1)_z^2} + \overline{Z_{(j_1)_z^2}}) + |Z_{(j_1)_z}|^2 + Z_{\omega_1}Z_{\phi_1}Z_{\tilde{\phi}_1}\\
&\quad + Z_{\phi_1}Z_{\tilde{\phi}_1}\left(Z_{(j_1)_z} + \overline{Z_{(j_1)_z}}\right) + Z_{\phi_1^2}Z_{\tilde{\phi}_1^2} + Z_{\omega_2} + (Z_{(j_2)_z} + \overline{Z_{(j_2)_z}}) + Z_{\phi_2}Z_{\tilde{\phi}_1} + Z_{\phi_1}Z_{\tilde{\phi}_2}\Big].
\end{aligned} \tag{4.54}$$

Now, let us go on to the representations with three boxes in the Young tableaux. For the primary operators $(\yng(3), \yng(1))$, $(\yng(2,1), \yng(1))$, and $(\yng(1,1,1), \yng(1))$, their contribution to the partition function decomposes as

$$\begin{aligned}
&\lim_{N\to\infty}\left(|\chi_{(\yng(3),\yng(1))}|^2 + |\chi_{(\yng(2,1),\yng(1))}|^2 + |\chi_{(\yng(1,1,1),\yng(1))}|^2\right)\\
&= Z_{hs}\Big[Z_{\phi_1}Z_{\phi_2} + \left(Z_{\omega_1} + Z_{(j_1)_z} + \overline{Z_{(j_1)_z}}\right)Z_{\phi_1^2} + Z_{\tilde{\phi}_1}Z_{\phi_1^3}\Big].
\end{aligned} \tag{4.55}$$

For the primary operators $(\yng(3), \yng(2))$, $(\yng(2,1), \yng(2))$, $(\yng(1,1,1), \yng(2))$, $(\yng(3), \yng(1,1))$, $(\yng(2,1), \yng(1,1))$, and $(\yng(1,1,1), \yng(1,1))$, their contribution to the partition function decomposes as

$$\begin{aligned}
&\lim_{N\to\infty}\left(|\chi_{(\yng(3),\yng(2))}|^2 + |\chi_{(\yng(2,1),\yng(2))}|^2 + |\chi_{(\yng(1,1,1),\yng(2))}|^2 + |\chi_{(\yng(3),\yng(1,1))}|^2 + |\chi_{(\yng(2,1),\yng(1,1))}|^2 + |\chi_{(\yng(1,1,1),\yng(1,1))}|^2\right)\\
&= Z_{hs}\Big[\left(Z_{\omega_2} + Z_{(j_2)_z} + \overline{Z_{(j_2)_z}}\right)Z_{\phi_1} + \left(Z_{\omega_1} + Z_{(j_1)_z} + \overline{Z_{(j_1)_z}}\right)Z_{\phi_2} + Z_{\phi_1^2}Z_{\tilde{\phi}_2} + Z_{\phi_1}Z_{\tilde{\phi}_1}Z_{\phi_2}\\
&\quad + \left(Z_{\omega_1^2} + Z_{\omega_1}Z_{(j_1)_z} + Z_{\omega_1}\overline{Z_{(j_1)_z}} + Z_{(j_1)_z^2} + \overline{Z_{(j_1)_z^2}} + Z_{(j_1)_z}\overline{Z_{(j_1)_z}}\right)Z_{\phi_1}\\
&\quad + \left(Z_{\omega_1} + Z_{(j_1)_z} + \overline{Z_{(j_1)_z}}\right)Z_{\phi_1^2}Z_{\tilde{\phi}_1} + Z_{\phi_1^3}Z_{\tilde{\phi}_1^2} + Z_{\phi_3}\Big].
\end{aligned} \tag{4.56}$$

The contribution from the primary operators $(\yng(3),\yng(3))$, $(\yng(2,1),\yng(3))$, $(\yng(1,1,1),\yng(3))$, $(\yng(3),\yng(2,1))$, $(\yng(2,1),\yng(2,1))$, $(\yng(1,1,1),\yng(2,1))$, $(\yng(3),\yng(1,1,1))$, $(\yng(2,1),\yng(1,1,1))$, and $(\yng(1,1,1),\yng(1,1,1))$, to the partition function decomposes as

$$
\begin{aligned}
&\lim_{N\to\infty}\Big(|\chi_{(\yng(3),\yng(3))}|^2+|\chi_{(\yng(2,1),\yng(3))}|^2+|\chi_{(\yng(1,1,1),\yng(3))}|^2+|\chi_{(\yng(3),\yng(2,1))}|^2\\
&\quad+|\chi_{(\yng(2,1),\yng(2,1))}|^2+|\chi_{(\yng(1,1,1),\yng(2,1))}|^2+|\chi_{(\yng(3),\yng(1,1,1))}|^2+|\chi_{(\yng(2,1),\yng(1,1,1))}|^2+|\chi_{(\yng(1,1,1),\yng(1,1,1))}|^2\Big)\\
&=Z_{hs}\Big[Z_{\omega_1^3}+Z_{\omega_1^2}\left(Z_{(j_1)_z}+\overline{Z_{(j_1)_z}}\right)+Z_{\omega_1}\left(Z_{(j_1)_z^2}+\overline{Z_{(j_1)_z^2}}+Z_{(j_1)_z}\overline{Z_{(j_1)_z}}\right)\\
&\quad+\left(Z_{(j_1)_z^3}+\overline{Z_{(j_1)_z^3}}+Z_{(j_1)_z^2}\overline{Z_{(j_1)_z}}+Z_{(j_1)_z}\overline{Z_{(j_1)_z^2}}\right)\\
&\quad+\Big(Z_{\omega_1^2}+Z_{\omega_1}\left(Z_{(j_1)_z}+\overline{Z_{(j_1)_z}}\right)+\left(Z_{(j_1)_z^2}+\overline{Z_{(j_1)_z^2}}+Z_{(j_1)_z}\overline{Z_{(j_1)_z}}\right)\Big)Z_{\phi_1}Z_{\tilde\phi_1}\\
&\quad+\left(Z_{\omega_1}+\left(Z_{(j_1)_z}+\overline{Z_{(j_1)_z}}\right)\right)Z_{\phi_1^2}Z_{\tilde\phi_1^2}+Z_{\phi_1^3}Z_{\tilde\phi_1^3}+Z_{\omega_1}Z_{\omega_2}\\
&\quad+Z_{\omega_1}\left(Z_{(j_2)_z}+\overline{Z_{(j_2)_z}}\right)+Z_{\omega_2}\left(Z_{(j_1)_z}+\overline{Z_{(j_1)_z}}\right)+\left(Z_{(j_1)_z}+\overline{Z_{(j_1)_z}}\right)\left(Z_{(j_2)_z}+\overline{Z_{(j_2)_z}}\right)\\
&\quad+\left(Z_{\omega_1}+Z_{(j_1)_z}+\overline{Z_{(j_1)_z}}\right)\left(Z_{\phi_1}Z_{\tilde\phi_2}+Z_{\phi_2}Z_{\tilde\phi_1}\right)+\left(Z_{\omega_2}+Z_{(j_2)_z}+\overline{Z_{(j_2)_z}}\right)Z_{\phi_1}Z_{\tilde\phi_1}\\
&\quad+Z_{\phi_1^2}Z_{\tilde\phi_1}Z_{\tilde\phi_2}+Z_{\phi_1}Z_{\tilde\phi_1^2}Z_{\phi_2}+Z_{\omega_1}+Z_{(j_1)_z}+\overline{Z_{(j_1)_z}}+Z_{\phi_1}Z_{\tilde\phi_3}+Z_{\phi_2}Z_{\tilde\phi_2}+Z_{\phi_3}Z_{\tilde\phi_1}\Big].
\end{aligned}
\tag{4.57}
$$

4.8 Interactions and a semi-local bulk theory

The three-point functions[3] involving the hidden symmetry currents amount to the following assignment of gauge generators T_n associated to the currents $j_n^{(s)}(z)$, which act on the states $|\phi_m\rangle$ and $|\tilde\phi_m\rangle$. We use the ket notation here, rather than the primary fields themselves, because while ϕ_m and $\tilde\phi_m$ have different scaling dimensions at infinite N, they are dual to scalar fields of the same mass that transform into one another under the hidden

[3] Some three-point functions are computed, and a general form of such three-point functions are postulated in Appendix 4.E.

gauge symmetries.

$$
\begin{aligned}
&T_n|\phi_m\rangle = |\phi_{n+m}\rangle, \quad T_n|\bar{\phi}_m\rangle = -|\bar{\phi}_{m-n}\rangle \ \ (n<m) \ \text{ or } \ -|\tilde{\phi}_{n-m+1}\rangle \ \ (n\geq m), \\
&T_n|\tilde{\phi}_m\rangle = -|\tilde{\phi}_{n+m}\rangle, \quad T_n|\bar{\tilde{\phi}}_m\rangle = |\bar{\tilde{\phi}}_{m-n}\rangle \ \ (n<m) \ \text{ or } |\phi_{n-m+1}\rangle \ \ (n\geq m).
\end{aligned}
\tag{4.58}
$$

Let us define the fields φ_r and $\tilde{\varphi}_r$ for $r \in \mathbb{Z}+\frac{1}{2}$ by

$$
\begin{aligned}
\varphi_r &= \phi_{r+\frac{1}{2}}, \quad \varphi_{-r} = \bar{\tilde{\phi}}_{r+\frac{1}{2}}, \\
\tilde{\varphi}_r &= \tilde{\phi}_{r+\frac{1}{2}}, \quad \tilde{\varphi}_{-r} = \bar{\phi}_{r+\frac{1}{2}}.
\end{aligned}
\tag{4.59}
$$

They are related by complex conjugation:

$$
\bar{\varphi}_r = \tilde{\varphi}_{-r}, \quad \bar{\tilde{\varphi}}_r = \varphi_{-r}. \tag{4.60}
$$

In terms of φ_r and $\tilde{\varphi}_r$, the gauge generators act as

$$
T_n|\varphi_r\rangle = |\varphi_{r+n}\rangle, \quad T_n|\tilde{\varphi}_r\rangle = -|\tilde{\varphi}_{r+n}\rangle. \tag{4.61}
$$

We also have

$$
\overline{T}_n|\varphi_r\rangle = -|\varphi_{r-n}\rangle, \quad \overline{T}_n|\tilde{\varphi}_r\rangle = |\tilde{\varphi}_{r-n}\rangle. \tag{4.62}
$$

which suggests the definition $T_{-n} = -\overline{T}_n$, or $j^{(s)}_{-n} = -\bar{j}^{(s)}_n$. Now (4.61) is extended to all $n \in \mathbb{Z}$. The action of T_n can be diagonalized by the Fourier transform:

$$
|\varphi(x)\rangle = \sum_{r\in\mathbb{Z}+1/2} e^{irx}|\varphi_r\rangle, \quad |\tilde{\varphi}(x)\rangle = \sum_{r\in\mathbb{Z}+1/2} e^{irx}|\tilde{\varphi}_r\rangle, \quad T(x) = \sum_{n\in\mathbb{Z}} e^{inx}T_n, \tag{4.63}
$$

where x is an auxiliary generating parameter. Here we also included the generator T_0 which assigns charge $+1$ to φ and charge -1 to $\bar{\varphi}$. With this definition, $|\bar{\varphi}(x)\rangle = |\tilde{\varphi}(x)\rangle, \overline{T}(x) = -T(x)$. We have

$$
T(x)|\varphi(y)\rangle = \delta(x-y)|\varphi(y)\rangle. \tag{4.64}
$$

Here x, y are understood to be periodically valued with periodicity 2π.

What is the interpretation of this result? We see that there is a circle worth of gauge generators $T(x)$, each of which corresponds to a tower of gauge fields in AdS$_3$, of spin $s = 1, 2, 3, \cdots, \infty$. Furthermore, these gauge generators commute, indicating Vasiliev theory with $U(1)^\infty$ "Chan-Paton factor". At the level of bulk equation of motion, we expect the infinite family of Vasiliev theories to decouple. They only interact through the AdS$_3$ boundary conditions that mix the matter scalar fields. The boundary condition is such that the "right moving" modes of $\varphi(x)$ on the circle, namely φ_r with $r > 0$ ($r = \frac{1}{2}, \frac{3}{2}, \cdots$) are dual to operators of dimension $\Delta_+ = 1 + \lambda$, whereas φ_r with $r < 0$ are dual to operators of dimension $\Delta_- = 1 - \lambda$. As a consequence of this boundary condition, the corresponding generating operator $\varphi(x; z, \bar{z})$ in the CFT has two-point function

$$\langle \varphi(x; z, \bar{z}) \bar{\varphi}(y; 0) \rangle = \sum_{r,s \in \mathbb{Z}+1/2} e^{irx+isy} \langle \varphi_r(z) \tilde{\varphi}_s(0) \rangle = \left(\frac{1}{|z|^{2+2\lambda}} - \frac{1}{|z|^{2-2\lambda}} \right) \frac{i}{2 \sin \frac{x-y}{2}} \tag{4.65}$$

in the large N limit.

Note that the spin-1 gauge field is included here. It is also natural to include the massless scalar ω_n, of spin $s = 0$. $|\varphi(x)\rangle$ labels a complex massive scalar in AdS$_3$, for each x. This spectrum precisely fits into Vasiliev's system in three dimensions. In earlier works, we did not consider the spin-1 gauge field in Vasiliev theory, because it is governed by $U(1) \times U(1)$ Chern-Simons action and would decouple from the higher spin gravity if it weren't for the matter scalar field. It is possible to choose the boundary condition on the spin-1 Chern-Simons gauge field in AdS$_3$ so that there is no dual spin-1 current in the boundary CFT. This is presumably why the spin-1 current $j_0^{(1)}(z)$ is missing from the spectrum of W_N minimal model. But the spin-1 currents $j_n^{(1)}(z)$ do exist in the infinite N limit. Usually, in three-dimensional Vasiliev theory, there is no propagating massless scalar field either. There is however, an auxiliary scalar field C_{aux} [10], whose equation of motion at the linearized level takes the form $\nabla_\mu C_{aux} = 0$. Classically, we could trade this equation with the massless Klein-Gordon equation $\Box C_{aux} = 0$, together with the $\Delta = 0$ boundary condition which eliminates normalizable finite energy states of this field in AdS_3. If this scalar field acquires a small mass, of order $1/N$ due to quantum corrections, then the boundary condition would allow for a normalizable state in AdS_3 of very small energy/conformal weight. We believe that this is the origin of the elementary light scalars ω_n themselves, in the infinite family of Vasiliev systems parameterized by the circle.

The identification of the single-trace operators, dual to elementary particles in the bulk, makes sense a priori only in the infinite N limit. Non-perturbatively, or at finite N, k, the infinite family $\phi_n, \tilde{\phi}_n, \omega_n, j_n^{(s)}$ should be cut off to a finite family. Due to the restrictions on the unitary representations of $SU(N)$ current algebra at level k or $k+1$, we expect the subscript n which counts the number of boxes in the Young tableau in the construction of the single-trace primaries to be cut off at $n \sim k$. This means that the circle that parameterize a continuous family of Vasiliev theories in AdS_3 should be rendered discrete, with spacing $\sim 2\pi/k$.

4.9 Discussion

We have proposed that the holographic dual of W_N minimal model in the 't Hooft limit, $k, N \to \infty$, $0 < \lambda < 1$, is a circle worth of Vasiliev theories in AdS_3 that couple with one another only through the boundary conditions on the matter scalars, which break all but one single tower of higher spin symmetries. It would seem to be a natural question to ask what is the CFT dual to the bulk theory with symmetry-preserving boundary conditions, that assign say the same scaling dimension Δ_+ to all matter scalars. If we are to flip the boundary condition on $\tilde{\phi}_n$, on the CFT side this corresponds to turning on the double trace deformation by $\tilde{\phi}_n\bar{\tilde{\phi}}_n$ and flow to the critical point (IR in this case). This deformation decreases the central charge $c \approx N(1-\lambda^2)$ by an order N^0 amount. It is unclear what is the fixed point one ends up with by turning on double trace deformations $\tilde{\phi}_n\bar{\tilde{\phi}}_n$ for all n (which should be cut off at $\sim k$), if there is such a nontrivial critical point at all.

There has been an alternative proposal on the holographic dual of W_N minimal model [42, 53, 54], as Vasiliev theory based on $hs[N] \simeq sl(N)$ higher spin algebra, with families of conical deficit solutions included to account for the primaries missing from the perturbative spectrum of Vasiliev theory. On the face of it, this proposal involves an entirely different limit, where N is held fixed, and an analytic continuation is performed in k so that the central charge c is large. The resulting CFT is not unitary. Furthermore, it is unclear to us that the analog of large N (or rather, large c) factorization holds in this limit, which would be necessary for the holographic dual to be weakly coupled.

There is also an intriguing parallel between the 't Hooft limit of W_N minimal model in two dimensions and Chern-Simons vector model in three dimensions. While the gauge invariant local operators and their correlation functions on $\mathbb{R}^3$ or S^3 in the three dimensional Chern-Simons vector model are expected to be computed by the parity violating Vasiliev theory in AdS_4 to all order in $1/N$, the duality in its naive form is not expected to hold for the CFT on three-manifolds of nontrivial topology (e.g. when the spatial manifold is a torus or a higher genus surface). This is because the topological degrees of freedom of the Chern-Simons gauge fields cannot be captured by a semi-classical theory in the bulk with Newton's constant that scales like $1/N$ rather than $1/N^2$. In a similar manner, the W_N minimal model CFT on $\mathbb{R}^2$ or S^2 in the large N admits a closed subsector, generated by the OPEs of the primary ϕ_1 along with higher spin currents, that is conjectured to be perturbatively dual to Vasiliev theory in AdS_3. This duality makes sense only perturbatively in $1/N$. The light primaries which in a sense arise from twistor sectors must be included to ensure that the CFT is modular invariant. Here we see that the bulk theory should be extended as well, to an infinite family of Vasiliev theories. It would be interesting to understand the analogous statement in the $\mathrm{AdS}_4/\mathrm{CFT}_3$ example, where the connection to ordinary string theory is better understood [52] .

4.A Higher spin charges

The higher spin charges of primary operators can be computed using the Coulomb gas formalism reviewed in [31, 55, 50]. In Coulomb gas formalism, the higher spin currents $W^{(s)}$ are functions of derivatives of the compact boson X, which can be constructed as follows. Considering the order-N differential operator $\mathcal{D}_N$ given by

$$(2iv_0)^N \mathcal{D}_N =: \prod_{i=1}^{N} (2iv_0\partial + h_i \cdot \partial X) : . \tag{4.66}$$

A tower of quasi-primary spin-s current $U^{(s)}$ is given by the coefficients of the expansion of D_N in the variable v_0,

$$\mathcal{D}_N = \partial^N + \sum_{s=1}^{N} (2iv_0)^{-k} U^{(s)} \partial^{N-s}. \tag{4.67}$$

For example, we have $U^{(1)} = 0$ and $U^{(2)} = -\frac{1}{2} : \partial X \cdot \partial X : +2v_0\rho \cdot \partial^2 X$, which is the stress tensor. The primary spin-s current $W^{(s)}$ can be constructed by taking linear combinations of derivatives of $U^{(s)}$, for example [56]:

$$\begin{aligned}
W^{(2)} &= U^{(2)}, \\
W^{(3)} &= U^{(3)} - \frac{N-2}{2}(2iv_0)\partial U^{(2)}, \\
W^{(4)} &= U^{(4)} - \frac{N-3}{2}(2iv_0)\partial U^{(3)} + \frac{(N-2)(N-3)}{10}(2iv_0)^2\partial^2 U^{(2)} \\
&\quad - \frac{(N-2)(N-3)(5N+7)}{10N^2(N^2-1)} : (U^{(2)})^2 :, \\
W^{(5)} &= U^{(5)} - \frac{N-4}{2}(2iv_0)\partial U^{(4)} + \frac{3(N-3)(N-4)}{28}(2iv_0)^2\partial^2 U^{(3)} \\
&\quad - \frac{(N-2)(N-3)(N-4)}{84}(2iv_0)^3\partial^3 U^{(2)} \\
&\quad + \frac{(N-3)(N-4)(7N+13)}{14N(N^2-1)}\Big((N-2)(2iv_0) : U^{(2)}\partial U^{(2)} : -2 : U^{(3)}U^{(2)} : \Big).
\end{aligned} \tag{4.68}$$

The higher spin charges w_k of the primary (Λ_+, Λ_-) are given by the eigenvalues of the zero modes of the higher spin currents $W^{(s)}$. The eigenvalues of the zero modes of the quasi-primaries $U^{(s)}$ are given by

$$u_s(v) = (-i)^{s-1} \sum_{i_1 < \cdots < i_s} \prod_{j=1}^{s} \left(v \cdot h_{i_j} + (s-j) v_0 \right). \tag{4.69}$$

where

$$v = \sqrt{\frac{p'}{p}} \Lambda_+ - \sqrt{\frac{p}{p'}} \Lambda_-, \qquad h_s = \omega_1 - \sum_{i=1}^{s-1} \alpha_i. \tag{4.70}$$

Plugging u_s into the formula (4.68), we obtain the higher spin charges w_s.

The higher spin charges of ϕ_1 and $\tilde{\phi}_1$ were computed in [11] in the large N limit. We generalize their formula to ϕ_n and $\tilde{\phi}_n$,

$$\begin{aligned} w_s(\phi_n) &= \frac{i^{s-2} \Gamma(s)^2 \Gamma(\lambda + s)}{\Gamma(2s-1) \Gamma(1+\lambda)}, \\ w_s(\tilde{\phi}_n) &= \frac{(-i)^{s-2} \Gamma(s)^2 \Gamma(\lambda + s)}{\Gamma(2s-1) \Gamma(1+\lambda)}. \end{aligned} \tag{4.71}$$

We check these two formulas up to $n = 2, s = 5$ using the above method.

We also propose that the higher spin charges of ω_n are given by n times the higher spin charges of ω_1 in the large N limit. For example, the higher spin charges of ω_n up to spin-5 are given by

$$\begin{aligned} w_2(\omega_n) &= \frac{n\lambda^2}{2N}, \\ w_3(\omega_n) &= i \frac{n\lambda^3}{6N}, \\ w_4(\omega_n) &= -\frac{n\lambda^2(1+\lambda^2)}{20N}, \\ w_5(\omega_n) &= -i \frac{n\lambda^3(5+\lambda^2)}{70N}. \end{aligned} \tag{4.72}$$

The above formulas are checked up to $n = 3$.

4.B An approximately conserved spin-2 current

The approximately conserved spin-2 field takes the form as

$$(j_1^{(2)})_z = \alpha \left(W_{-1}^{(3)} - \frac{3}{2} i\lambda L_{-1} \right) L_{-1}\omega_1 = \alpha (W_{-2}^{(3)} - i\lambda \partial^2)\omega_1, \tag{4.73}$$

where α is a normalization constant. We check that this is a Virasoro primary operator:

$$\begin{aligned} L_1(W_{-2}^{(3)} - i\lambda\partial^2)\omega_1 &= \left[4W_{-1}^{(3)} - 2i\lambda L_{-1} \right] \omega_1 = 0, \\ L_2(W_{-2}^{(3)} - i\lambda\partial^2)\omega_1 &= (6w_3 - 6ih\lambda)\omega_1 = 0, \end{aligned} \tag{4.74}$$

where we have used the null-state equation for ω_1:

$$W_{-1}^{(3)}\omega_1 = \frac{i\lambda}{2}\partial\omega_1. \tag{4.75}$$

Let us compute the normalization constant α. Considering the three-point function $\Big\langle W(z)\bar{\omega}_1(z_1)(W_{-2}^{(3)} - i\lambda$
since it is a three-point function of three conformal primaries, it takes the form as

$$\left\langle W(z)\bar{\omega}_1(z_1)(W_{-2}^{(3)} - i\lambda\partial^2)\omega_1(z_2) \right\rangle = \frac{a_1}{(z-z_1)(z-z_2)^5(z_1-z_2)^{-1}}. \tag{4.76}$$

The structure constant a_1 can be determined by performing contour integral $\oint_{z_2} dz(z-z_2)^4$ on the both hand side. On RHS, we obtain

$$\oint_{z_2} dz \frac{a_1}{(z-z_1)(z-z_2)(z_1-z_2)^{-1}} = -a_1 \tag{4.77}$$

On LHS, we have

$$\begin{aligned} \left\langle \bar{\omega}_1(z_1) W_2^{(3)}(W_{-2}^{(3)} - i\lambda\partial^2)\omega_1(z_2) \right\rangle &= \left\langle \bar{\omega}_1(z_1) \left(8W_0^{(4)} + \frac{4}{5}(\lambda^2 - 4)L_0 - 12i\lambda W_0^{(3)} \right) \omega_1(z_2) \right\rangle \\ &= -\frac{2\lambda^2(1-\lambda^2)}{N}. \end{aligned} \tag{4.78}$$

Now, we perform a contour integral $\int_{z_1} dz$ on (4.76), we obtain

$$\begin{aligned}\left\langle W^{(3)}_{-2}\bar{\omega}_1(z_1)(W^{(3)}_{-2} - i\lambda\partial^2)\omega_1(z_2)\right\rangle &= \oint_{z_1} dz \frac{a_1}{(z-z_1)(z-z_2)^5(z_1-z_2)^{-1}} \\ &= \frac{a_1}{(z_1-z_2)^4} = \frac{2\lambda^2(1-\lambda^2)}{N}\frac{1}{(z_1-z_2)^4}.\end{aligned} \tag{4.79}$$

Using similar method, we obtain

$$\begin{aligned}\left\langle \bar{\omega}_1(z_1)W^{(3)}_{-2}\omega_1(z_2)\right\rangle &= \oint_{z_2} dz \frac{b}{(z-z_1)^3(z-z_2)^3(z_1-z_2)^{-3}} \\ &= \frac{6b}{(z_2-z_1)^5(z_1-z_2)^{-3}} = \frac{i\lambda^3}{N}\frac{1}{(z_1-z_2)^2}.\end{aligned} \tag{4.80}$$

We have

$$\left\langle (W^{(3)}_{-2} + i\lambda\partial^2)\bar{\omega}_1(z_1)(W^{(3)}_{-2} - i\lambda\partial^2)\omega_1(z_2)\right\rangle = \frac{2\lambda^2(1-\lambda^2)}{N}\frac{1}{(z_1-z_2)^4}. \tag{4.81}$$

The normalization constant α is

$$\alpha = \sqrt{\frac{N}{2\lambda^2(1-\lambda^2)}}. \tag{4.82}$$

4.C Null-state equations

The $W_\infty[\lambda]$, in the $c \to \infty$ limit, and truncating to the generators $W^{(s)}_n$, $|n| < s$, reduces to the wedge algebra $hs(\lambda)$, which is given by

$$[W^{(s)}_m, W^{(t)}_n] = \sum_{u=2,4,6,\cdots}^{s+t-|s-t|-1} g^{st}_u(m,n;\lambda) W^{(s+t-u)}_{m+n}, \tag{4.83}$$

where the structure constant $g^{st}_u(m,n;\lambda)$ is

$$g^{st}_u(m,n;\lambda) = \frac{q^{u-2}}{2(u-1)!}\phi^{st}_u(\lambda)N^{st}(m,n), \tag{4.84}$$

and

$$N^{st}_u(\lambda) = \sum_{k=0}^{u-1} \frac{(-1)^k\Gamma(u)\Gamma(s-m)\Gamma(s+m)\Gamma(t-n)\Gamma(t+n)}{\Gamma(1+k)\Gamma(u-k)\Gamma(s-m-k)\Gamma(s+m+k-u+1)\Gamma(t+n-k)\Gamma(t-n+k-u+1)}, \tag{4.85}$$

and

$$\phi_u^{st}(\lambda) = {}_4F_3\left[\begin{matrix} \frac{1}{2}+\lambda\ ,\ \frac{1}{2}-\lambda\ ,\ \frac{2-u}{2}\ ,\ \frac{1-u}{2} \\ \frac{3}{2}-s\ ,\ \ \frac{3}{2}-t\ ,\ \ \frac{1}{2}+s+t-u \end{matrix}\middle| 1\right]. \tag{4.86}$$

q is an arbitrary constant controls the normalization of the higher spin generators. In our convention, $q = i/4$. Using this commutator (4.83), we can derived a set of null-state equations for ϕ_n, $\tilde{\phi}_n$ and ω_n.

Consider a primary operator $\mathcal{O}$. The descendants of $\mathcal{O}$ can be separated into two classes. The descendants in the first class are the operators take the form as a combination of $W^{(s)}_{-n}$, $0 < n < s$, acting on $\mathcal{O}$. The rest of the descendants are in the second class. The descendants in the first class have the norm of order one, and the descendants in the second class have the norm of order N. The bulk dual of the descendants in the first class are single- or multi-particle states without boundary higher spin gauge field excitation, and the bulk dual descendants in the second class are the states with boundary higher spin gauge field excitations. Now, let us focus on the primary $(\Box, 0)$. The partition of $(\Box, 0)$, after modding out the contribution from the boundary higher spin gauge field, takes the form as

$$\begin{aligned}\lim_{N\to\infty} Z_{hs}^{-1}|\chi_{(\Box,0)}|^2 = Z_{\phi_1} &= \frac{q^{\frac{1+\lambda}{2}}\bar{q}^{\frac{1+\lambda}{2}}}{(1-q)(1-\bar{q})} \\ &= q^{\frac{1+\lambda}{2}}\bar{q}^{\frac{1+\lambda}{2}}(1+q+q^2+\cdots)(1+\bar{q}+\bar{q}^2+\cdots).\end{aligned} \tag{4.87}$$

This means that at each level (m, n), there is only one independent descendent in the first class. Therefore, the Gram matrices

$$\begin{pmatrix} [L_1^n, L_{-1}^n], [W_n^{(s)}, L_{-1}^n] \\ [L_1^n, W_{-n}^{(s)}], [W_n^{(s)}, W_{-n}^{(s)}] \end{pmatrix}, \tag{4.88}$$

for $0 < n < s$, are rank 1, and have a singular vector, which gives the null-state equation:

$$W^{(s)}_{-n}\phi_1 = \frac{i^{s-2}\Gamma(s)\Gamma(n+s)\Gamma(s+\lambda)}{\Gamma(n+1)\Gamma(2s-1)\Gamma(n+\lambda+1)}\partial^n\phi_1. \tag{4.89}$$

Similarly, we have the null-state equation for $\tilde{\phi}_1$:

$$W^{(s)}_{-n}\tilde{\phi}_1 = \frac{(-i)^{s-2}\Gamma(s)\Gamma(n+s)\Gamma(s-\lambda)}{\Gamma(n+1)\Gamma(2s-1)\Gamma(n-\lambda+1)}\partial^n\tilde{\phi}_1. \tag{4.90}$$

One can then express the operators $W^{(s)}_{-n}\phi_1, W^{(s)}_{-n}\tilde{\phi}_1$ as $\partial^m\bar{\partial}^n\phi_1, \bar{\partial}^n\phi_1$ for $0 < n < s$.

Next, let us consider the operator $(\square,\square)$. After moving out the contribution of boundary higher spin gauge fields, the partition function of $(\square,\square)$ takes the form as

$$\lim_{N\to\infty} Z^{-1}_{hs}|\chi_{(\square,\square)}|^2 = (1+q+2q^2+3q^3+4q^4+\cdots)(1+\bar{q}+2\bar{q}^2+3\bar{q}^3+4\bar{q}^4+\cdots). \tag{4.91}$$

At level one, there is one descendent in the first class. The Gram matrix

$$\begin{pmatrix} [L_1, L_{-1}], [W^{(s)}_1, L_{-1}] \\ [L_1, W^{(s)}_{-1}], [W^{(s)}_1, W^{(s)}_{-1}] \end{pmatrix}, \tag{4.92}$$

is rank one, and gives the null-state equations:

$$W^{(s)}_{-1}\omega_1 = \frac{sw_s}{2h}\partial\omega_1. \tag{4.93}$$

For $s = 3, 4, 5$, we have

$$\begin{aligned} W^{(3)}_{-1}\omega_1 &= i\frac{\lambda}{2}\partial\omega_1, \\ W^{(4)}_{-1}\omega_1 &= -\frac{1+\lambda^2}{5}\partial\omega_1, \\ W^{(5)}_{-1}\omega_1 &= -i\frac{\lambda(5+\lambda^2)}{14}\partial\omega_1. \end{aligned} \tag{4.94}$$

At level two, there are two descendants in the first class. The Gram matrix

$$\begin{pmatrix} [L_1^2, L_{-1}^2], [W^{(3)}_2, L_{-1}^2], [W^{(s)}_2, L_{-1}^2] \\ [L_1^2, W^{(3)}_{-2}], [W^{(3)}_2, W^{(3)}_{-2}], [W^{(s)}_2, W^{(3)}_{-2}] \\ [L_1^2, W^{(s)}_{-2}], [W^{(3)}_2, W^{(s)}_{-2}], [W^{(s)}_2, W^{(s)}_{-2}] \end{pmatrix}, \tag{4.95}$$

has one singular vector. For $s = 4$, this gives the null state equation

$$W^{(4)}_{-2}\omega_1 = -\frac{1}{2}\partial^2\omega_1 + i\frac{\lambda}{2}W^{(3)}_{-2}\omega_1. \tag{4.96}$$

In general, at level n, there are n independent descendants in the first class. They can be written as $\partial^{n-1} j_1^{(1)}$, $\partial^{n-2} j_1^{(2)}$, $\cdots$, and $j_1^{(n)}$, or equivalently $\partial^n \omega_1$, $\partial^{n-1} W_{-2}^{(3)} \omega_1$, $\cdots$, and $W_{-n}^{(n+1)} \omega_1$. All the other descendants are related to them by null state equations.

4.D W_N characters

As reviewed in [31, 50], the characters of the primary operators in the W_N minimal model are given by the formula

$$\chi_{(\Lambda_+,\Lambda_-)} = \frac{1}{\eta(\tau)^{N-1}} \sum_{w \in W, n \in \Gamma_{pp'}} \epsilon(w) q^{\frac{1}{2}|w(\lambda)+\lambda'+n|^2 + \frac{c}{24}} \tag{4.97}$$

where $p = k + N$, $p' = k + N + 1$, W is the Weyl group, $\Gamma_{pp'}$ is $\sqrt{pp'}$ times the root lattice Λ_{root}, and λ, λ' are

$$\lambda = \sqrt{\frac{p'}{p}}(\Lambda_+ + \rho), \quad \lambda' = -\sqrt{\frac{p}{p'}}(\Lambda_- + \rho). \tag{4.98}$$

In the large N limit, the terms with nonzero n in the summation over the lattice $\Gamma_{pp'}$ are of order $\mathcal{O}(q^N)$, and can be ignored. By evaluating the formula (4.97), we obtain the following characters:

$$
\begin{aligned}
\chi_{(0,0)} &= 1 + q^2 + 2q^3 + \cdots \\
\chi_{(\ydiagram{1},0)} &= q^{h(\ydiagram{1},0)} \left(1 + q + 2q^2 + 4q^3 + \cdots\right) \\
\chi_{(\ydiagram{1,1},0)} &= q^{h(\ydiagram{1,1},0)} \left(1 + q + 3q^2 + 5q^3 + \cdots\right) \\
\chi_{(\ydiagram{2},0)} &= q^{h(\ydiagram{2},0)} \left(1 + q + 3q^2 + 5q^3 + \cdots\right) \\
\chi_{(\ydiagram{1,1,1},0)} &= q^{h(\ydiagram{1,1,1},0)} \left(1 + q + 3q^2 + 6q^3 + \cdots\right) \\
\chi_{(\ydiagram{2,1},0)} &= q^{h(\ydiagram{2,1},0)} \left(1 + 2q + 4q^2 + 9q^3 + \cdots\right) \\
\chi_{(\ydiagram{3},0)} &= q^{h(\ydiagram{3},0)} \left(1 + q + 3q^2 + 6q^3 + \cdots\right) \\
\chi_{(\ydiagram{1},\ydiagram{1})} &= q^{h(\ydiagram{1},\ydiagram{1})} \left(1 + q + 3q^2 + 6q^3 + \cdots\right) \\
\chi_{(\ydiagram{2},\ydiagram{2})} &= q^{h(\ydiagram{2},\ydiagram{2})} \left(1 + q + 3q^2 + 6q^3 + \cdots\right) \\
\chi_{(\ydiagram{1,1},\ydiagram{1,1})} &= q^{h(\ydiagram{1,1},\ydiagram{1,1})} \left(1 + q + 4q^2 + 8q^3 + \cdots\right) \\
\chi_{(\ydiagram{3},\ydiagram{3})} &= q^{h(\ydiagram{3},\ydiagram{3})} \left(1 + q + 3q^2 + 6q^3 + \cdots\right) \\
\chi_{(\ydiagram{2,1},\ydiagram{2,1})} &= q^{h(\ydiagram{2,1},\ydiagram{2,1})} \left(1 + 2q + 6q^2 + 14q^3 + \cdots\right) \\
\chi_{(\ydiagram{1,1,1},\ydiagram{1,1,1})} &= q^{h(\ydiagram{1,1,1},\ydiagram{1,1,1})} \left(1 + q + 4q^2 + 9q^3 + \cdots\right) \\
\chi_{(\ydiagram{2},\ydiagram{1})} &= q^{h(\ydiagram{2},\ydiagram{1})} \left(1 + q + 3q^2 + 6q^3 + \cdots\right) \\
\chi_{(\ydiagram{1,1},\ydiagram{1})} &= q^{h(\ydiagram{1,1},\ydiagram{1})} \left(1 + 2q + 4q^2 + 9q^3 + \cdots\right) \\
\chi_{(\ydiagram{3},\ydiagram{2})} &= q^{h(\ydiagram{3},\ydiagram{2})} \left(1 + q + 3q^2 + 6q^3 + \cdots\right) \\
\chi_{(\ydiagram{2,1},\ydiagram{2})} &= q^{h(\ydiagram{2,1},\ydiagram{2})} \left(1 + 2q + 5q^2 + 11q^3 + \cdots\right) \\
\chi_{(\ydiagram{2,1},\ydiagram{1,1})} &= q^{h(\ydiagram{2,1},\ydiagram{1,1})} \left(1 + 2q + 5q^2 + 12q^3 + \cdots\right) \\
\chi_{(\ydiagram{1,1,1},\ydiagram{1,1})} &= q^{h(\ydiagram{1,1,1},\ydiagram{1,1})} \left(1 + 2q + 5q^2 + 11q^3 + \cdots\right) \\
\chi_{(\ydiagram{2,1},\ydiagram{1})} &= q^{h(\ydiagram{2,1},\ydiagram{1})} \left(1 + 2q + 5q^2 + 11q^3 + \cdots\right)
\end{aligned}
\tag{4.99}
$$

$$
\begin{aligned}
\chi_{(\ydiagram{1,1,1},\ydiagram{1})} &= q^{h(\ydiagram{1,1,1},\ydiagram{1})}\left(1+2q+5q^2+10q^3+\cdots\right)\\
\chi_{(\ydiagram{3},\ydiagram{1})} &= q^{h(\ydiagram{3},\ydiagram{1})}\left(1+q+3q^2+6q^3+\cdots\right)\\
\chi_{(\ydiagram{2},\ydiagram{1,1})} &= q^{h(\ydiagram{2},\ydiagram{1,1})}\left(1+2q+5q^2+10q^3+\cdots\right)\\
\chi_{(\ydiagram{1,1},\ydiagram{2})} &= q^{h(\ydiagram{1,1},\ydiagram{2})}\left(1+2q+5q^2+10q^3+\cdots\right)\\
\chi_{(\ydiagram{1,1,1},\ydiagram{2})} &= q^{h(\ydiagram{1,1,1},\ydiagram{2})}\left(1+2q+6q^2+12q^3+\cdots\right)\\
\chi_{(\ydiagram{3},\ydiagram{1,1})} &= q^{h(\ydiagram{3},\ydiagram{1,1})}\left(1+2q+5q^2+11q^3+\cdots\right)\\
\chi_{(\ydiagram{3},\ydiagram{2,1})} &= q^{h(\ydiagram{3},\ydiagram{2,1})}\left(1+2q+5q^2+11q^3+\cdots\right)\\
\chi_{(\ydiagram{2,1},\ydiagram{1,1,1})} &= q^{h(\ydiagram{2,1},\ydiagram{1,1,1})}\left(1+3q+7q^2+17q^3+\cdots\right)\\
\chi_{(\ydiagram{3},\ydiagram{1,1,1})} &= q^{h(\ydiagram{3},\ydiagram{1,1,1})}\left(1+2q+6q^2+13q^3+\cdots\right)\\
\chi_{(\ydiagram{2,1},\ydiagram{3})} &= q^{h(\ydiagram{2,1},\ydiagram{3})}\left(1+2q+5q^2+11q^3+\cdots\right)\\
\chi_{(\ydiagram{1,1,1},\ydiagram{2,1})} &= q^{h(\ydiagram{1,1,1},\ydiagram{2,1})}\left(1+3q+7q^2+17q^3+\cdots\right)\\
\chi_{(\ydiagram{1,1,1},\ydiagram{3})} &= q^{h(\ydiagram{1,1,1},\ydiagram{3})}\left(1+2q+6q^2+13q^3+\cdots\right)
\end{aligned}
\tag{4.100}
$$

4.E Some three-point functions

In this section, we will compute several three-point functions involving the approximately conserved spin-1 current $(j_n^{(1)})_z$ in the large N limit. For simplicity, we will suppress the index z in the following discussion. Let us first consider the three-point functions of the form $\left\langle j_n^{(1)}\bar{\phi}_m\bar{\bar{\phi}}_{n-m+1}\right\rangle$. They are given by taking a derivative on the three-point function $\left\langle \omega_n\bar{\phi}_m\bar{\bar{\phi}}_{n-m+1}\right\rangle$. For example, by taking one derivative on

$$
\left\langle \omega_1(z_1)\bar{\phi}_1(z_2)\bar{\bar{\phi}}_1(z_3)\right\rangle = \frac{1}{N}\frac{1}{|z_{12}|^{2\lambda}|z_{23}|^2|z_{13}|^{-2\lambda}}, \tag{4.101}
$$

we obtian

$$\left\langle j_1^{(1)}(z_1)\bar{\phi}_1(z_2)\bar{\bar{\phi}}_1(z_3)\right\rangle = \frac{1}{\sqrt{N}}\frac{1}{|z_{12}|^{2\lambda}|z_{23}|^2|z_{13}|^{-2\lambda}}\left(\frac{1}{z_{13}}-\frac{1}{z_{12}}\right). \tag{4.102}$$

Similarly by taking a derivative on (4.14) and (4.19), we obtain

$$\begin{aligned}&\left\langle j_2^{(1)}(z_1)\bar{\phi}_1(z_2)\bar{\bar{\phi}}_2(z_3)\right\rangle = \left\langle j_2^{(1)}(z_1)\bar{\phi}_2(z_2)\bar{\bar{\phi}}_1(z_3)\right\rangle = \left\langle j_3^{(1)}(z_1)\bar{\phi}_1(z_2)\bar{\bar{\phi}}_3(z_3)\right\rangle = \left\langle j_3^{(1)}(z_1)\bar{\phi}_3(z_2)\bar{\bar{\phi}}_1(z_3)\right\rangle\\ &= \frac{1}{\sqrt{N}}\frac{1}{|z_{12}|^{2\lambda}|z_{23}|^2|z_{13}|^{-2\lambda}}\left(\frac{1}{z_{13}}-\frac{1}{z_{12}}\right).\end{aligned} \tag{4.103}$$

We postulate the general form of the three-point function to be

$$\left\langle j_n^{(1)}(z_1)\bar{\phi}_m(z_2)\bar{\bar{\phi}}_{n-m+1}(z_3)\right\rangle = \frac{1}{\sqrt{N}}\frac{1}{|z_{12}|^{2\lambda}|z_{23}|^2|z_{13}|^{-2\lambda}}\left(\frac{1}{z_{13}}-\frac{1}{z_{12}}\right). \tag{4.104}$$

Next, let us consider the three-point function of the form $\left\langle j_n^{(1)}\phi_m\bar{\phi}_{n+m}\right\rangle$ and $\left\langle j_n^{(1)}\tilde{\phi}_m\bar{\bar{\phi}}_{n+m}\right\rangle$. The computation of this three-point function is a bit subtle. Let us first show an example $\left\langle j_1^{(1)}(z_1)\phi_1(z_2)\bar{\phi}_2(z_3)\right\rangle$. To compute this three-point function, we consider the three-point functions:

$$\begin{aligned}\langle\omega_1(z_1)\phi_1(z_2)(\overline{\square\square},\overline{\square})(z_3)\rangle &= \frac{1}{\sqrt{2}}\frac{1}{|z_{23}|^{2h_{(\square,0)}+2h_{(\overline{\square\square},\overline{\square})}-2h_{(\square,\square)}}|z_{12}|^{2h_{(\square,0)}+2h_{(\square,\square)}-2h_{(\overline{\square\square},\overline{\square})}}|z_{13}|^{2h_{(\overline{\square\square},\overline{\square})}+2h_{(\square,\square)}-2h_{(\square,0)}}},\\ \left\langle\omega_1(z_1)\phi_1(z_2)(\overline{\substack{\square\\ \square}},\overline{\square})(z_3)\right\rangle &= \frac{1}{\sqrt{2}}\frac{1}{|z_{23}|^{2h_{(\square,0)}+2h_{(\overline{\substack{\square\\ \square}},\overline{\square})}-2h_{(\square,\square)}}|z_{12}|^{2h_{(\square,0)}+2h_{(\square,\square)}-2h_{(\overline{\substack{\square\\ \square}},\overline{\square})}}|z_{13}|^{2h_{(\overline{\substack{\square\\ \square}},\overline{\square})}+2h_{(\square,\square)}-2h_{(\square,0)}}}.\end{aligned} \tag{4.105}$$

By taking the derivative ∂_{z_1} and taking the large N limit, we obtain

$$\begin{aligned}\langle\partial\omega_1(z_1)\phi_1(z_2)(\overline{\square\square},\overline{\square})(z_3)\rangle &= \frac{1}{\sqrt{2}}\frac{1}{|z_{23}|^{2(1-\lambda)}}\left(\frac{\lambda}{N}\frac{1}{z_{12}}-\frac{\lambda+\lambda^2}{N}\frac{1}{z_{12}}\right),\\ \left\langle\partial\omega_1(z_1)\phi_1(z_2)(\overline{\substack{\square\\ \square}},\overline{\square})(z_3)\right\rangle &= \frac{1}{\sqrt{2}}\frac{1}{|z_{23}|^{2(1-\lambda)}}\left(-\frac{\lambda}{N}\frac{1}{z_{12}}+\frac{\lambda-\lambda^2}{N}\frac{1}{z_{12}}\right).\end{aligned} \tag{4.106}$$

Taking the difference of these two three-point functions, we obtain

$$\left\langle j_1^{(1)}(z_1)\phi_1(z_2)\bar{\phi}_2(z_3)\right\rangle = \frac{1}{\sqrt{N}}\frac{1}{|z_{23}|^{2(1+\lambda)}}\left(\frac{1}{z_{12}}-\frac{1}{z_{13}}\right). \tag{4.107}$$

In a similar way, we also compute the three-point functions

$$\left\langle j_1^{(1)}(z_1)\phi_2(z_2)\bar{\phi}_3(z_3)\right\rangle = \left\langle j_2^{(1)}(z_1)\phi_1(z_2)\bar{\phi}_3(z_3)\right\rangle = \frac{1}{\sqrt{N}}\frac{1}{|z_{23}|^{2(1+\lambda)}}\left(\frac{1}{z_{12}} - \frac{1}{z_{13}}\right), \tag{4.108}$$

and also

$$\begin{aligned}&\left\langle j_1^{(1)}(z_1)\tilde{\phi}_1(z_2)\bar{\tilde{\phi}}_2(z_3)\right\rangle = \left\langle j_1^{(1)}(z_1)\tilde{\phi}_2(z_2)\bar{\tilde{\phi}}_3(z_3)\right\rangle = \left\langle j_2^{(1)}(z_1)\tilde{\phi}_1(z_2)\bar{\tilde{\phi}}_3(z_3)\right\rangle \\ &= -\frac{1}{\sqrt{N}}\frac{1}{|z_{23}|^{2(1-\lambda)}}\left(\frac{1}{z_{12}} - \frac{1}{z_{13}}\right).\end{aligned} \tag{4.109}$$

We postulate the general form of these kind of three-point functions to be

$$\begin{aligned}\left\langle j_n^{(1)}(z_1)\phi_m(z_2)\bar{\phi}_{n+m}(z_3)\right\rangle &= \frac{1}{\sqrt{N}}\frac{1}{|z_{23}|^{2(1+\lambda)}}\left(\frac{1}{z_{12}} - \frac{1}{z_{13}}\right), \\ \left\langle j_n^{(1)}(z_1)\tilde{\phi}_m(z_2)\bar{\tilde{\phi}}_{n+m}(z_3)\right\rangle &= -\frac{1}{\sqrt{N}}\frac{1}{|z_{23}|^{2(1-\lambda)}}\left(\frac{1}{z_{12}} - \frac{1}{z_{13}}\right).\end{aligned} \tag{4.110}$$

Part II

AdS$_4$ higher spin holography

Chapter 5

ABJ Triality: from Higher Spin Fields to Strings

5.1 Introduction and Summary

It has long been speculated that the tensionless limit of string theory is a theory of higher spin gauge fields. One of the most important explicit and nontrivial construction of interacting higher spin gauge theory is Vasiliev's system in AdS_4 [57, 58, 22]. It was conjectured by Klebanov and Polyakov [19], and by Sezgin and Sundell [20, 59], that the parity invariant A-type and B-type Vasiliev theories are dual to 2+1 dimensional bosonic and fermionic $O(N)$ or $U(N)$ vector models in the singlet sector. Substantial evidence for these conjectures has been provided by comparison of three-point functions [33, 34], and analysis of higher spin symmetries [60, 23, 37, 51].

It was noted in [61, 21] that, at large N, the free $O(N)$ and $U(N)$ theories described above each have a family of one parameter conformal deformations, corresponding to turning on a finite Chern-Simons level for the $O(N)$ or $U(N)$ gauge group. It was conjectured in [21] that the bulk duals of the resultant Chern-Simons vector models is given by a one parameter family of parity violating Vasiliev theories. In the bulk description parity is broken by a nontrivial phase in function f in Vasiliev's theory that controls bulk interactions. This conjecture appeared to pass some nontrivial checks [21] but also faced some puzzling challenges [21]. In this paper we will find significant additional evidence in support of the proposal of [21] from the study of the bulk duals of supersymmetric vector Chern-Simons theories.

The duality between Vasiliev theory and 3d Chern-Simons boundary field theories does not rely on supersymmetry, and, indeed, most studies of this duality have been carried out in the non-supersymmetric context. However it is possible to construct supersymmetric analogues of the Type A and type B bosonic Vasiliev theories [58, 22, 59, 62, 63]. With appropriate boundary conditions, these supersymmetric Vasiliev theories preserve all higher spin symmetries and are conjectured to be dual to free boundary supersymmetric gauge theories. In the spirit of [21] it is natural to attempt to construct bulk duals of the one parameter set of interacting supersymmetric Chern-Simons vector theories obtained by turning on a finite level k for the Chern-Simons terms (recall that Chern Simons coupled gauge fields are free only in the limit $k \to \infty$). Interacting supersymmetric Chern-Simons theories differ from their free counterparts in three ways. First, as emphasized above, their Chern-Simons level is taken to be finite. According to the conjecture of [21] this is accounted for by turning on the appropriate phase in Vasiliev's equations. Second the Lagrangian includes potential terms of the schematic form ϕ^6 and Yukawa terms of the schematic form $\phi^2\psi^2$, where ϕ and ψ are fundamental and antifundamental scalars and fermions in the field theory.

These terms may be regarded as double and triple trace deformations of the field theory; as is well known, the effect of such terms on the dual bulk theory may be accounted for by an appropriate modification of boundary conditions [64]. Lastly, supersymmetric field theories with $\mathcal{N} = 4$ and $\mathcal{N} = 6$ supersymmetry necessarily have two gauge groups with matter in the bifundamental. Such theories may be obtained by from theories with a single Chern-Simons coupled gauge group at level k and fundamental matter by gauging a global symmetry with Chern-Simons level $-k$. In the dual bulk theory this gauging is implemented by a modification of the boundary conditions of the bulk vector gauge field [65].

These elements together suggest that it should be possible to find one parameter families of Vasiliev theories that preserve some supersymmetry upon turning on the parity violating bulk phase, if and only if one also modifies the boundary conditions of all bulk scalars, fermions and sometimes gauge fields in a coordinated way. In this paper we find that this is indeed the case. We are able to formulate one parameter families of parity violating Vasiliev theory (enhanced with Chan-Paton factors, see below) that preserve $\mathcal{N} = 0, 1, 2, 3, 4$ or 6 supersymmetries depending on boundary conditions. In every case we identify conjectured dual Chern-Simons vector models dual to our bulk constructions.[1]

The identification of parity violating Vasiliev theory with prescribed boundary conditions as the dual of Chern-Simons vector models pass a number of highly nontrivial checks. By considering of boundary conditions alone, we will be able to determine the exact relation between the parity breaking phase θ_0 of Vasiliev theory, and two and three point function coefficients of Chern-Simons vector models at large N. These imply non-perturbative rela-

[1] A similar analysis of the breaking of higher spin symmetry by boundary conditions allows us to demonstrate that all deformations of type A or type B Vasiliev theories break all higher spin symmetries other than the conformal symmetry. We are also able to use this analysis to determine the functional form of the double trace part of higher spin currents that contain a scalar field.

tions among purely field theoretic quantities that are previously unknown (and presumably possible to prove by generalizing the computation of correlators in Chern-Simons-scalar vector model of [66] using Schwinger-Dyson [2] equations to the supersymmetric theories). The results also agree with the relation between θ_0 and Chern-Simons 't Hooft coupling $\lambda = N/k$ determined in [21] by explicit perturbative computations at one-loop and two-loop order.

From a physical viewpoint, the most interesting Vasiliev theory presented in this paper is the $\mathcal{N} = 6$ theory. It was already suggested in [21] that a supersymmetric version of the parity breaking Vasiliev theory in AdS_4 should be dual to the vector model limit of the $\mathcal{N} = 6$ ABJ theory, that is, a $U(N)_k \times U(M)_{-k}$ Chern-Simons-matter theory in the limit of large N, k but finite M. Since the ABJ theory is also dual to type IIA string theory in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ with flat B-field, it was speculated that the Vasiliev theory must therefore be a limit of this string theory. The concrete supersymmetric $\mathcal{N} = 6$ Vasiliev system presented in this paper allows us to turn the suggestion of [21] into a precise conjecture for a duality between three distinct theories that are autonomously well defined at least in particular limits.

The $\mathcal{N} = 6$ Vasiliev theory, conjectured below to be dual to $U(N) \times U(M)$ ABJ theory has many elements absent in more familiar bosonic Vasiliev systems. First theory is 'supersymmetric' in the bulk. This means that all fields of the theory are functions of fermionic variables ψ_i $(i = 1 \ldots 6)$ which obey Clifford algebra commutation relations $\{\psi_i, \psi_j\} = 2\delta_{ij}$ (all bulk fields are also functions of the physical spacetime variables x_μ $(\mu = 1 \ldots 4)$ as well as Vasiliev's twistor variables y_α, z_α, $\bar{y}_{\dot{\alpha}}$, $\bar{z}_{\dot{\beta}}$, as in bosonic Vasiliev theory). Next the star product used in the bulk equations is the usual Vasiliev star product times matrix multi-

[2]See [21] for these equations in the Chern-Simons fermion model.

plication in an auxiliary $M \times M$ space. The physical effect of this maneuver is to endow the bulk theory with a $U(M)$ gauge symmetry under which all bulk fields transform in the adjoint. Finally, for the reasons described above, interactions of the theory are also modified by a bulk phase, and bulk scalars, fermions and gauge fields obey nontrivial boundary conditions that depend on this phase.

The triality between $U(N) \times U(M)$ ABJ theory, type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, and supersymmetric parity breaking Vasiliev theory may qualitatively be understood in the following manner. The propagating degrees of freedom of ABJ theory consist of bifundamental fields that we denote by A_i and antibifundamental fields that we will call B_i. A basis for the gauge singlet operators of the theory is given by the traces $\mathrm{Tr}(A_1B_1A_2B_2 \ldots A_mB_m)$. As is well known from the study of ABJ duality, these single trace operators are dual to single string states. The basic 'partons' (the A and B fields) out of which this trace is composed are held together in this string state by the 'glue' of $U(N)$ and $U(M)$ gauge interactions.

Let us now study the limit $M \ll N$. In this limit the glue that joins B type fields to A type fields (provided by the gauge group $U(M)$) is significantly weaker than the glue that joins A fields to B fields (this glue is supplied by $U(N)$ interactions). In this limit the trace effectively breaks up into m weakly interacting particles A_1B_1, A_2B_2 ... A_mB_m. These particles, which transform in the adjoint of $U(M)$, are the dual to the $U(M)$ adjoint fields of the dual $\mathcal{N} = 6$ Vasiliev theory. Indeed the spectrum of operators of field theory operators of the form AB precisely matches the spectrum of bulk fields of the dual Vasiliev system.

If our picture is correct, the fields of Vasiliev's theory must bind together to make up fundamental IIA strings as M/N is increased. We now describe a qualitative way in which this might happen. The bulk Vasiliev theory has gauge coupling $g \sim 1/\sqrt{N}$, It follows that the *bulk* 't Hooft coupling is $\lambda_{bulk} = g^2 M \sim M/N$. In the limit $M/N \ll 1$, the bulk Vasiliev theory is effectively weakly coupled. As M/N increases, a class of multi-particle states of higher spin fields acquire large binding energies due to interactions, and are mapped to the single closed string states in type IIA string theory. Roughly speaking, the fundamental string of string theory is simply the flux tube string of the non abelian bulk Vasiliev theory.

Note that although we claim a family of supersymmetric Vasiliev theory with Chan-Paton factors and certain prescribed boundary conditions is equivalent to string theory on AdS$_4$, we are *not* suggesting that Vasiliev's equations are the same as the corresponding limit of closed string field equations. Not all single closed string states are mapped to single higher spin particles; infact the only closed strings that are mapped to Vasiliev's particles are those dual to the operators of the form TrAB. Closed string field theory is the weakly interacting theory of the 'glueball' bound states of the Vasiliev fields; it is not a weakly interacting description of Vasiliev's fields themselves.

Let us note a curious aspect of the conjectured duality between Vasiliev's theory and ABJ theory. The gauge groups $U(N)$ and $U(M)$ appear on an even footing in the ABJ field theory. In the bulk Vasiliev description, however, the two gauge groups play a very different role. The gauge group $U(M)$ is manifest as a gauge symmetry in the bulk. However $U(N)$ symmetry is not manifest in the bulk (just as the $U(N)$ symmetry is not manifest in the bulk dual of $\mathcal{N} = 4$ Yang Mills); the dynamics of this gauge group that leads to the emergence of the background spacetime for Vasiliev theory. The deconfinement transition for $U(M)$ is simply a deconfinement transition of the adjoint bulk degrees of freedom, while the deconfinement transition for $U(N)$ is associated with the very different process of 'black hole formation'. If our proposal for the dual description is correct, the gauged Vasiliev theory must enjoy an $N \leftrightarrow M$ symmetry, which, from the bulk viewpoint is a sort of level – rank duality. Of course even a precise statement for the claim of such a level rank duality only makes sense if Vasiliev theory is well defined 'quantum mechanically' (i.e. away from small $\frac{M}{N}$) at least in the large N limit.

We have noted above that Vasiliev's theory should not be identified with closed string field theory. There may, however, be a sense in which it might be thought of as an open string field theory. We use the fact that there is an alternative way to engineer Chern-Simons vector models using string theory [67], that is by adding N_f D6-branes wrapped on $\mathrm{AdS}_4 \times \mathbb{RP}^3$ inside the $\mathrm{AdS}_4 \times \mathbb{CP}^3$, which preserves $\mathcal{N} = 3$ supersymmetry and amounts to adding fundamental hypermultiplets of the $U(N)_k$ Chern-Simons gauge group. In the "minimal radius" limit where we send M to zero, with flat B-field flux $\frac{1}{2\pi\alpha}\int_{\mathbb{CP}^1} B = \frac{N}{k} + \frac{1}{2}$, the geometry is entirely supported by the N_f D6-branes [68].[3] This type IIA open+closed string theory is dual to $\mathcal{N} = 3$ Chern-Simons vector model with N_f hypermultiplet flavors. The duality suggests that the open+closed string field theory of the D6-branes reduces to precisely a supersymmetric Vasiliev theory in the minimal radius limit. Note that unlike the ABJ triality, here the open string fields on the D6-branes and the nonabelian higher spin gauge fields in Vasiliev's system both carry $U(N_f)$ Chan-Paton factors, and we expect one-to-one correspondence between single open string states and single higher spin particle states.

[3]We thank Daniel Jafferis for making this important suggestion and O. Aharony for related discussions.

5.2 Vasiliev's higher spin gauge theory in AdS$_4$ and its supersymmetric extension

The Vasiliev systems that we study in this paper are defined by a set of bulk equations of motion together with boundary conditions on the bulk fields. In this section we review the structure of the bulk equations. We turn to the consideration of boundary conditions in the next section.

In this section we first present a detailed review of bulk equations of the 'standard' Vasiliev theory. We then describe nonabelian and supersymmetric extensions of these equations. Throughout this paper we work with the so-called non-minimal version of Vasiliev's equations, which describe the interactions of a field of *each* non-negative integer spin s in AdS$_4$. Under the AdS/CFT correspondence non-minimal Vasiliev equations are conjectured to be dual to gauged $U(N)$ Chern-Simons-matter boundary theories.[4]

There are exactly two 'standard' non-minimal Vasiliev theories that preserve parity symmetry. These are the type A/B theories, which are conjectured to be dual to bosonic/fermionic $SU(N)$ vector models, restricted to the $SU(N)$-singlet sector. Parity invariant Vasiliev theories are particular examples of a larger class of generically parity violating Vasiliev theories. These theories appear to be labeled by a real even function of one real variable. In Section 5.2.1 we present a review of these theories. It was conjectured in [21] that a class of these parity violating theories are dual to $SU(N)$ Chern-Simons vector models.

In Section 5.2.2 we then present a straightforward nonabelian extension of Vasiliev's

[4] The non minimal equations admit a consistent truncation to the so-called minimal version of Vasiliev's equations; this truncation projects out the gauge fields for odd spins and are conjectured to supply the dual to $SO(N)$ Chern-Simons boundary theories.

system, by introducing $U(M)$ Chan-Paton factors into Vasiliev's star product. The result of this extension is to promote the bulk gauge field to a $U(M)$ gauge field; all other bulk fields transform in the adjoint of $U(M)$. The local gauge transformation parameter of Vasiliev's theory is also promoted to a local $M \times M$ matrix field that transforms in the adjoint of $U(M)$. The nature of the boundary CFT dual to the non abelian Vasiliev theory depends on boundary conditions. With 'standard' magnetic type boundary conditions for all gauge fields (that set prescribed values for the field strengths restricted to the boundary) the dual boundary CFT is obtained simply by coupling M copies of (otherwise non interacting) matter multiplets to the same boundary Chern-Simons gauge field. The boundary theory has a 'flavour' $U(M)$ global symmetry that acts on the M identical matter multiplets.

In Section 5.2.3 we then introduce the so called n-extended supersymmetric Vasiliev theory (generalizing the special cases studied earlier in [57, 58, 22, 59, 62]). The main idea is to enhance Vasiliev's fields to functions of n fermionic fields ψ_i ($i = 1 \ldots n$; we assume n to be even) which obey a Clifford algebra[5]. This extension promotes the usual Vasiliev's fields to $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ dimensional matrices (or operators) that act on the $2^{\frac{n}{2}}$ dimensional representation of the Clifford algebra. The local Vasiliev gauge transformations are also promoted to functions of ψ_i, and so $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ matrices or operators[6]. Half of the resultant fields (and gauge transformations) are fermionic; the other half are bosonic.

[5]We emphasize that n should not be confused with the number of globally conserved supercharges $4\mathcal{N}$ (equivalently $4\mathcal{N}$ is the number of supercharges in the superconformal algebra of the dual three-dimensional CFT). n characterizes only the local structure of Vasiliev's equations of motion. $\mathcal{N}$ on the other hand depends on the choice of boundary condition for bulk fields of spin 0, 1/2 and 1. As we will see $\mathcal{N} \leq 6$ for parity violating Vasiliev theories, as expected from the dual CFT_3 (n, or course, can be arbitrarily large).

[6]The bulk equations of motion for n extended supersymmetric Vasiliev theory is identical to those for $n = 2$ theory extended by $U(2^{\frac{n}{2}-1})$ Chan Paton factors. However, the language of n extended supersymmetric Vasiliev theory is more convenient when the boundary conditions of the problem break part of this $U(2^{\frac{n}{2}-1})$ symmetry, as will be the case later in this paper.

5.2.1 The standard parity violating bosonic Vasiliev theory

In this section we present the 'standard' non minimal Vasiliev equations, allowing, however, for parity violation.

Coordinates

In Euclidean space the fields of Vasiliev's theory are functions of a collection of bosonic variables $(x, Y, Z) = (x^\mu, y^\alpha, \bar{y}^{\dot{\alpha}}, z^\alpha, \bar{z}^{\dot{\alpha}})$. x^μ $(\mu = 1 \ldots 4)$ are an arbitrary set of coordinates on the four dimensional spacetime manifold. y^α and z^α are spinors under $SU(2)_L$ while $\bar{y}^{\dot{\alpha}}$ and $\bar{z}^{\dot{\alpha}}$ are spinors under a separate $SU(2)_R$. As we will see below, Vasiliev's equations enjoy invariance under *local* (in spacetime) $SO(4) = SU(2)_L \times SU(2)_R$ rotations of y^α, z^α, $\bar{y}^{\dot{\alpha}}$ and $\bar{z}^{\dot{\alpha}}$. This local $SO(4)$ rotational invariance, which, as we will see below is closely related to the tangent space symmetry of the first order formulation of general relativity, is only a small part of the much larger gauge symmetry of Vasiliev's theory.

Star Product

Vasiliev's equations are formulated in terms of a star product. This is just the usual local product in coordinate space; whereas in auxiliary space it is given by

$$\begin{aligned}
&f(Y,Z) * g(Y,Z) \\
&= f(Y,Z) \exp\left[\epsilon^{\alpha\beta}\left(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha}\right)\left(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta}\right) + \epsilon^{\dot{\alpha}\dot{\beta}}\left(\overleftarrow{\partial}_{y^{\dot{\alpha}}} + \overleftarrow{\partial}_{z^{\dot{\alpha}}}\right)\left(\overrightarrow{\partial}_{y^{\dot{\beta}}} - \overrightarrow{\partial}_{z^{\dot{\beta}}}\right)\right] g(Y,Z) \\
&= \int d^2u d^2v d^2\bar{u} d^2\bar{v} e^{u^\alpha v_\alpha + \bar{u}^{\dot{\alpha}}\bar{v}_{\dot{\alpha}}} f(y+u, \bar{y}+\bar{u}, z+u, \bar{z}+\bar{u}) g(y+v, \bar{y}+\bar{v}, z-v, \bar{z}-\bar{v}).
\end{aligned} \tag{5.1}$$

In the last line, the integral representation of the star product is defined by the contour for (u^α, v^α) along $e^{\pi i/4}\mathbb{R}$ in the complex plane, and $(\bar{u}^{\dot{\alpha}}, \bar{v}^{\dot{\alpha}})$ along the contour $e^{-\pi i/4}\mathbb{R}$. It is obvious from the first line of (5.1) that $1 * f = f * 1 = f$; this fact may be used to set the normalization of the integration measure in the second line. The star product is associative but non commutative; in fact it may be shown to be isomorphic to the usual Moyal star product under an appropriate change of variables. In Appendix 5.A.1 we describe our conventions for lowering spinor indices and present some simple identities involving the star product.

Below we will make extensive use of the so called Kleinian operators K and $\overline{K}$ defined as

$$K = e^{z^\alpha y_\alpha}, \qquad \overline{K} = e^{\bar{z}^{\dot\alpha} \bar{y}_{\dot\alpha}}. \tag{5.2}$$

They have the property (see Appendix 5.A.1 for a proof)

$$\begin{aligned} &K * K = \overline{K} * \overline{K} = 1, \\ &K * f(y, z, \bar{y}, \bar{z}) * K = f(-y, -z, \bar{y}, \bar{z}), \quad \overline{K} * f(y, z, \bar{y}, \bar{z}) * \overline{K} = f(y, z, -\bar{y}, -\bar{z}). \end{aligned} \tag{5.3}$$

Master fields

Vasiliev's master fields consists of an x-space 1-form

$$W = W_\mu dx^\mu,$$

a Z-space 1-form

$$S = S_\alpha dz^\alpha + S_{\dot\alpha} d\bar{z}^{\dot\alpha},$$

and a scalar B, all of which depend on spacetime as well as the internal twistor coordinates which we denote collectively as $(x, Y, Z) = (x^\mu, y^\alpha, \bar{y}^{\dot\alpha}, z^\alpha, \bar{z}^{\dot\alpha})$. It is sometimes convenient to write W and S together as a 1-form on (x, Z)-space

$$\mathcal{A} = W + S = W_\mu dx^\mu + S_\alpha dz^\alpha + S_{\dot\alpha} d\bar{z}^{\dot\alpha}.$$

$\mathcal{A}$ will be regarded as a gauge connection with respect to the $*$-algebra.

We also define

$$\hat{S} = S - \frac{1}{2}z_\alpha dz^\alpha - \frac{1}{2}\bar{z}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}},$$
$$\hat{\mathcal{A}} = W + \hat{S} = \mathcal{A} - \frac{1}{2}z_\alpha dz^\alpha - \frac{1}{2}\bar{z}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}} = W_\mu dx^\mu + (-\frac{1}{2}z_\alpha + S_\alpha)dz^\alpha + (-\frac{1}{2}\bar{z}_{\dot{\alpha}} + S_{\dot{\alpha}})d\bar{z}^{\dot{\alpha}}. \tag{5.4}$$

Let d_x be the exterior derivative with respect to the spacetime coordinates x^μ and denote by d_Z the exterior derivative with respect to the twistor variables $(z^\alpha, \bar{z}^{\dot{\alpha}})$. We will write $d = d_x + d_Z$. We will also find it useful to define the field strength

$$\begin{aligned} \mathcal{F} &= d_x\hat{\mathcal{A}} + \hat{\mathcal{A}} \star \hat{\mathcal{A}} \\ &= (d_x W + W * W) + \left(d_x\hat{S} + \{W, \hat{S}\}_*\right) + \left(\hat{S} * \hat{S}\right). \end{aligned} \tag{5.5}$$

Note also that

$$\hat{S} * \hat{S} = d_z S + S * S + \frac{1}{4}\left(\epsilon_{\alpha\beta} dz^\alpha dz^\beta + \epsilon_{\dot{\alpha}\dot{\beta}} d\bar{z}^{\dot{\alpha}} d\bar{z}^{\dot{\beta}}\right). \tag{5.6}$$

Gauge Transformations

Vasiliev's master fields transform under a large set of gauge symmetries. We will see later that the AdS$_4$ vacuum solution partially Higgs or breaks this gauge symmetry group down to a subgroup of large gauge transformations - either the higher spin symmetry group or the conformal group depending on boundary conditions.

Infinitesimal gauge transformations are generated by an arbitrary real function $\epsilon(x, Y, Z)$. By definition under gauge transformations

$$\begin{aligned} \delta\hat{\mathcal{A}} &= d_x\epsilon + \hat{\mathcal{A}} * \epsilon - \epsilon * \hat{\mathcal{A}}, \\ \delta B &= -\epsilon * B + B * \pi(\epsilon). \end{aligned} \tag{5.7}$$

In other words the 1-form master field transforms as a gauge connection under the star algebra while B transforms as a 'twisted' adjoint field. The operation π that appears in (5.7) is defined as follows

$$\pi\left(y, z, dz, \overline{y}, \overline{z}, d\overline{z}\right) = \left(-y, -z, -dz, \overline{y}, \overline{z}, d\overline{z}\right)$$

Since ϵ does not involve differentials in $(z, \bar{z})$, the action of π on ϵ is equivalent to conjugation by K, namely $\pi(\epsilon) = K * \epsilon * K$. ($\pi$ acting on a 1-form in $(z_\alpha, \bar{z}_{\dot{\alpha}})$ acts by conjugation by K together with flipping the sign of dz).

It follows from (5.7) that the field strength $\mathcal{F}$ (and so each of the three brackets on the RHS of the second line of (5.5)) transform in the adjoint. The same is true of $B * K$.

$$\begin{aligned} \delta\mathcal{F} &= [\mathcal{F}, \epsilon]_*, \\ \delta(B \star K) &= -\epsilon * (B * K) + (B * K) * \epsilon, \end{aligned} \tag{5.8}$$

When expanded in components the first line of (5.7) implies that

$$\begin{aligned} \delta W_\mu &= \partial_\mu \epsilon + W_\mu * \epsilon - \epsilon * W_\mu, \\ \delta\hat{S}_\alpha &= \hat{S}_\alpha * \epsilon - \epsilon * \hat{S}_\alpha. \end{aligned} \tag{5.9}$$

In terms of unhatted variables,

$$\begin{aligned} \delta\mathcal{A} &= d\epsilon + \mathcal{A} * \epsilon - \epsilon * \mathcal{A}, \\ \delta S_\alpha &= \frac{\partial\epsilon}{\partial z^\alpha} + S_\alpha * \epsilon - \epsilon * S_\alpha. \end{aligned} \tag{5.10}$$

Truncation

The following truncation is imposed on the master fields and gauge transformation parameter ϵ. Define

$$R = K\overline{K}.$$

We require

$$[R, W]_* = \{R, S\}_* = [R, B]_* = [R, \epsilon]_* = 0. \tag{5.11}$$

More explicitly, this is the statement that W_μ, B and ϵ are even functions of (Y, Z) whereas $S_\alpha, S_{\dot\alpha}$ are odd in (Y, Z),

$$\begin{aligned}
&W_\mu(x, y, \bar{y}, z, \bar{z}) = W_\mu(x, -y, -\bar{y}, -z, -\bar{z}),\\
&S_\alpha(x, y, \bar{y}, z, \bar{z}) = -S_\alpha(x, -y, -\bar{y}, -z, -\bar{z}),\\
&S_{\dot\alpha}(x, y, \bar{y}, z, \bar{z}) = -S_{\dot\alpha}(x, -y, -\bar{y}, -z, -\bar{z}),\\
&B(x, y, \bar{y}, z, \bar{z}) = B(x, -y, -\bar{y}, -z, -\bar{z}).\\
&\epsilon(x, y, \bar{y}, z, \bar{z}) = \epsilon(x, -y, -\bar{y}, -z, -\bar{z}).
\end{aligned} \tag{5.12}$$

A physical reason for the imposition of this truncation is the spin statistics theorem. As the physical fields of Vasiliev's theory are all commuting, they must also transform in the vector (rather than spinor) conjugacy class of the $SO(4)$ tangent group; the projection (5.12) ensures that this is the case. One might expect from this remark that the consistency of Vasiliev's equations requires this truncation; we will see explicitly below that this is the case.

Reality Conditions

It turns out that Vasiliev's master fields admit two consistent projections that may be used to reduce their number of degrees of freedom. These two projections are a generalized reality projection (somewhat analogous to the Majorana condition for spinors) and a so called 'minimal' truncation (very loosely analogous to a chirality truncation for spinors). These two truncations are defined in terms of two natural operations defined on the master field; complex conjugation and an operation defined by the symbol ι. In this subsection we first define these two operations, and then use them to define the generalized reality projection. We will also briefly mention the minimal projection, even though we will not use the later in this paper.

Vasiliev's fields master fields admit a straightforward complex conjugation operation, $\mathcal{A} \to \mathcal{A}^*$, defined by complex conjugating each of the component fields of Vasiliev theory and also the spinor variables Y, Z[7]

$$(y^\alpha)^* = \bar{y}_{\dot{\alpha}}, \quad (\bar{y}_{\dot{\alpha}})^* = y^\alpha, \quad (z^\alpha)^* = \bar{z}_{\dot{\alpha}}, \quad (\bar{z}_{\dot{\alpha}})^* = z^\alpha. \tag{5.13}$$

It is easily verified that

$$(M * N)^* = M^* * N^* \tag{5.14}$$

where M is an arbitrary p form and N and arbitrary q form. In other words complex conjugation commutes with the star and wedge product, without reversing the order of either of these products. Note also that the complex conjugation operation squares to the identity.

We now turn to the definition of the operation ι; this operation is defined by

$$\iota: \ (y, \bar{y}, z, \bar{z}, dz, d\bar{z}) \to (iy, i\bar{y}, -iz, -i\bar{z}, -idz, -id\bar{z}), \tag{5.15}$$

The signs in (5.15) are chosen[8] to ensure

$$\iota(f * g) = \iota(g) * \iota(f) \tag{5.16}$$

(see (5.186) for a proof). In other words ι reverses the order of the $*$ product. Note however that ι by definition does not affect the order of wedge products of forms. As a consequence

[7] As complex conjugation of $SO(3,1)$ interchanges left and right moving spinors, our definition of complex conjugation (the analytic continuation of the Lorentzian notion) must also have this property.

[8] Changing the RHS of (5.15) by an overall sign makes no difference to fields that obey (5.12).

ι picks up an extra minus sign when acting on the product of two one-forms

$$\iota(C * D) = -\iota(D) * \iota(C)$$

(see (5.187) for a proof; the same equation is true if C is a p form and D a q form provided p and q are both odd; if at least one of p and q is even we have no minus sign).

We now define the generalized reality projection that we will require Vasiliev's master fields to obey throughout this paper (this projection defines the non-minimal Vasiliev theory which we study through this paper). The projection is defined by the conditions

$$\iota(W)^* = -W, \quad \iota(S)^* = -S, \quad \iota(B)^* = \overline{K} * B * \overline{K} = K * B * K \tag{5.17}$$

The equality of the two different expressions supplied for $\iota(B)^*$ in (5.17) follows upon using the fact B commutes with $R = K\overline{K}$ (see (5.11)).

It is easily verified that (5.17) implies that

$$\iota\left(\mathcal{F}\right)^* = -\mathcal{F} \tag{5.18}$$

(see (5.192) for an expansion in components) and that

$$\iota(B * K)^* = B * \overline{K}, \quad \iota(B * \overline{K})^* = B * K. \tag{5.19}$$

(5.17) may be thought of as a combination of two separate projections. The first is the 'standard' reality projection (see (5.188)). The second is the 'minimal truncation'(5.189). As discussed in Appendix 5.A.3, it is consistent to simultaneously impose invariance of Vasiliev's master field under both these projections. This operation defines the minimal Vasiliev theory (dual to $SO(N)$ Chern-Simons field theories). We will not study the minimal theory in this paper.

Equations of motion

Vasiliev's gauge invariant equations of motion take the form

$$\begin{aligned}&\mathcal{F} = d_x\hat{\mathcal{A}} + \hat{\mathcal{A}} * \hat{\mathcal{A}} = f_*(B * K)dz^2 + \overline{f}_*(B * \overline{K})d\bar{z}^2,\\ &d_xB + \hat{\mathcal{A}} * B - B * \pi(\hat{\mathcal{A}}) = 0.\end{aligned} \tag{5.20}$$

where $f(X)$ is a holomorphic function of X, $\overline{f}$ its complex conjugate, and $f_*(X)$ the corresponding $*$-function of X. Namely, $f_*(X)$ is defined by replacing all products of X in the Taylor series of $f(X)$ by the corresponding star products.

Note that both sides of the first of (5.20) are gauge adjoints, while the second line of that equation transforms in the twisted adjoint. In Appendix 5.A.4 we have demonstrated that the second equation of (5.20) may be derived from the first (assuming that $f(X)$ is a non-degenerate function) using the Bianchi identity

$$d_x\mathcal{F} + [A, \mathcal{F}]_* = 0 \tag{5.21}$$

In Appendix 5.A.4 we have also expanded Vasiliev's equations in components to clarify their physical content. As elaborated in (5.193) and (5.194), it follows from (5.20) that the field strength $dW + W * W$ is flat and that the adjoint fields $B * K$, S_α and $S_{\dot\alpha}$ are covariantly constant. In addition, various components of these adjoint fields commute or anticommute with each other under the star product (see (5.203) for a listing). The fields $\hat{S}_\alpha$ and $\hat{S}_\beta$, however, fail to commute with each other; their commutation relations are given by

$$\begin{aligned}&[\hat{S}_\alpha, \hat{S}_\beta]_* = \epsilon_{\alpha\beta} f_*(B * K)\\ &[\hat{S}_{\dot\alpha}, \hat{S}_{\dot\beta}]_* = \epsilon_{\dot\alpha\dot\beta} \bar{f}_*(B * \bar{K})\end{aligned} \tag{5.22}$$

Using various formulae presented in the Appendix (see e.g. (5.190)) it is easily verified that the Vasiliev equations, (expanded in the Appendix as (5.193) and (5.194)) map to themselves under the reality projection (5.17). The same is true of the minimal truncation projection.

Equivalences from field redefinitions

Vasiliev's equations are characterized by a single complex holomorphic function f. In this subsection we address the following question: to what extent to different functions f label different theories?

Any field redefinition that preserves the gauge and Lorentz transformation properties of all fields, but changes the form of f clearly demonstrates an equivalence of the theories with the corresponding choices of f. The most general field redefinitions consistent with gauge and Lorentz transformations and the form of Vasiliev's equations are

$$
\begin{aligned}
&B \to g_*(B * K) * K \\
&\widehat{S}_z \equiv (-\frac{1}{2}z_\alpha + S_\alpha)dz^\alpha \to \widehat{S}_z * h_*(B * K), \\
&\widehat{S}_{\bar{z}} \equiv (-\frac{1}{2}\bar{z}_{\dot{\alpha}} + \bar{S}_{\dot{\alpha}}) * d\bar{z}^{\dot{\alpha}} \to \widehat{S}_{\bar{z}} * \tilde{h}_*(-B * \overline{K}).
\end{aligned}
\tag{5.23}
$$

Several comments are in order. First note that the field redefinitions above obviously preserve form structure and gauge transformations properties. In particular these redefinitions preserve the fact that $B * K$, S_z and $S_{\overline{z}}$ transform in the adjoint representation of the gauge group. Second the field redefinitions above are purely holomorphic (e.g. g_* is a function only of $B * K$ but not of $B * \overline{K}$). It is not difficult to convince oneself that this is necessary in order to preserve the holomorphic form of Vasiliev's equations. Finally we have chosen to multiply the redefined functions S_z and $S_{\overline{z}}$ with functions from the right rather than the left. There is no lack of generality in this, however, as

$$\begin{aligned}
&\widehat{S}_z * h_*(B * K) = h_*(-B * K) * \widehat{S}_z, \qquad \widehat{S}_z * \overline{h}_*(B * \overline{K}) = \overline{h}_*(B * \overline{K}) * \widehat{S}_z, \\
&\widehat{S}_{\overline{z}} * h_*(B * K) = h_*(B * K) * \widehat{S}_{\overline{z}}, \qquad \widehat{S}_{\overline{z}} * \overline{h}_*(B * \overline{K}) = \overline{h}_*(-B * \overline{K}) * \widehat{S}_{\overline{z}},
\end{aligned} \tag{5.24}$$

((5.24) follows immediately from (5.203) derived in the Appendix). Finally, we have inserted a minus sign into the argument of the function $\tilde{h}$ for future convenience.

The reality conditions (5.17) impose constraints on the functions g, h and $\tilde{h}$. It is not difficult to verify that g is forced to be an *odd real* function $g(X)$. $g(X)$ is forced to be odd because the complex conjugation operation turns K into $\overline{K}$. When g is odd, however, the truncation (5.11) may be used to turn $\overline{K}$ back into K. For instance, with $g_*(X) = g_1 X + g_3 X * X * X + \cdots$, the field redefinition is

$$B \to g_1 B + g_3 B * K * B * K * B + \cdots \tag{5.25}$$

The RHS is still real because $K * B * K = \overline{K} * B * \overline{K}$ (it would not be real if $g(X)$ were not odd).

In order to examine the constraints of (5.17) on the functions h and $\tilde{h}$ note that

$$\begin{aligned}
\iota(S_z * h(B * K) + S_{\overline{z}} * \tilde{h}_*(-B * \overline{K}))^* &= \overline{h}(B * \overline{K}) * (-S_{\overline{z}}) + \overline{\tilde{h}}(-B * K) * (-S_z) \\
&= -\left(S_{\overline{z}} * \overline{h}(-B * \overline{K}) + S_z * \overline{\tilde{h}}(B * K)\right)
\end{aligned} \tag{5.26}$$

(where in the last step we have used (5.24)). It follows that the redefined function $\widehat{S}$ obeys the reality condition (5.17) if and only if

$$\tilde{h} = \overline{h}$$

where $\overline{h}$ is the complex conjugate of the function h.

The effect of the field redefinition of B is simply to permit a redefinition of the argument of the function f in Vasiliev's equations by an arbitrary odd real function. The effect of the field redefinition of $\widehat{S}$ may be deduced as follows. The $dx^\mu \wedge dx^\nu$ component of Vasiliev's - the assertion that W is a flat connection (see (5.193)) - is clearly preserved by this field redefinition. The $dx \wedge dZ$ components of the equation asserts that $\widehat{S}_z$ and $\widehat{S}_{\overline{z}}$ are covariantly constant. As $B * K$ and $B * \overline{K}$ are also covariantly constant (see (5.194)) the redefinition (5.23) clearly preserves this equation as well. However the dZ^2 components of the equations become

$$\begin{aligned}
&\widehat{S}_z * h_*(B * K) * \widehat{S}_z * h_*(B * K) = f_*(B * K) dz^2, \\
&\left\{ \widehat{S} * h_*(B * K), \widehat{S}_{\overline{z}} * \overline{h}_*(-B * \overline{K}) \right\}_* = 0, \\
&\widehat{S}_{\overline{z}} * \overline{h}_*(-B * \overline{K}) * \widehat{S}_{\overline{z}} * \overline{h}_*(-B * \overline{K}) = \overline{f}_*(B * \overline{K}) d\bar{z}^2.
\end{aligned} \tag{5.27}$$

Using (5.24) and the fact that $B * K$ commutes with $B * \overline{K}$ (this is obvious as K and $\overline{K}$ commute), these equations may be recast as

$$\begin{aligned}
&h_*(-B * K) * \left(\widehat{S}_z * \widehat{S}_z \right) * h_*(B * K) = f_*(B * K) dz^2, \\
&h_*(-B * K) * \left(\left\{ \widehat{S}, \widehat{S}_{\overline{z}} \right\}_* \right) * \overline{h}_*(-B * \overline{K}) = 0, \\
&\overline{h}_*(B * \overline{K}) * \left(\widehat{S}_{\overline{z}} * \widehat{S}_{\overline{z}} \right) * \overline{h}_*(-B * \overline{K}) = \overline{f}_*(B * \overline{K}) d\bar{z}^2.
\end{aligned} \tag{5.28}$$

(5.28) is precisely the dZ^2 component of the Vasiliev equation (the third equation in (5.193)) with the replacement

$$f_*(X) \to h_*(-X)^{-1} * f_*(X) * h_*(X)^{-1}, \tag{5.29}$$

or simply $f(X) \to h(X)^{-1} h(-X)^{-1} f(X)$.

So we see that the theory is really defined by $f(X)$ up to a change of variable $X \to g(X)$ for some odd real function $g(X)$ and multiplication by an invertible holomorphic even function. Provided that the function $f(X)$ admits a power series expansion about $X = 0$ and that $f(0) \neq 0$,[9] in Appendix 5.A.6 we demonstrate that we can use these field redefinitions

[9]This condition can probably be weakened, but cannot be completely removed. For example if $f(X)$ is

to put $f(X)$ in the form

$$f(X) = \frac{1}{4} + X \exp(i\theta(X)) \tag{5.30}$$

where $\theta(X) = \theta_0 + \theta_2 X^2 + \cdots$ is an arbitrary real even function.

Ignoring the special cases for which $f(X)$ cannot be cast into the form (5.30), the function $\theta(X)$ determines the general parity-violating Vasiliev theory.

The AdS solution

While Vasiliev's system is formulated in terms of a set of background independent equations, the perturbation theory is defined by expanding around the AdS_4 vacuum. In order to study this solution it is useful to establish some conventions. Let e_0^a and w_0^{ab} $(a, b = 1 \ldots 4)$ denote the usual vielbein and spin connection one-forms on any space (the index a transforms under the vector representation of the tangent space $SO(4)$). We define the corresponding bi-spinor objects

$$e_{\alpha\dot\beta} = \frac{1}{4} e^a \sigma^a_{\alpha\dot\beta}, \qquad w_{\alpha\beta} = \frac{1}{16} w^{ab} \sigma^{ab}_{\alpha\beta}, \qquad w_{\dot\alpha\dot\beta} = -\frac{1}{16} w^{ab} \bar\sigma^{ab}_{\dot\alpha\dot\beta}. \tag{5.31}$$

(see Appendix 5.A.7 for definitions of the σ matrices that appear in this equation.) Let e_0 and ω_0 be the vielbein and spin connection of Euclidean AdS_4 with unit radius. It may be shown that (see Appendix 5.A.8 for some details)

$$\begin{aligned} \mathcal{A} &= W_0(x|Y) \equiv e_0(x|Y) + \omega_0(x|Y) \\ &= (e_0)_{\alpha\dot\beta} y^\alpha \bar y^{\dot\beta} + (\omega_0)_{\alpha\beta} y^\alpha y^\beta + (\omega_0)_{\dot\alpha\dot\beta} \bar y^{\dot\alpha} \bar y^{\dot\beta}, \quad B = 0. \end{aligned} \tag{5.32}$$

an odd function, it is easy to convince oneself that it cannot be cast into the form (5.30). In this paper we will be interested in the Vasiliev duals to field theories. In the free limit, the dual Vasiliev theories to the field theory in question are given by $f(X)$ of the form (5.30) with $\theta = 0$. It follows that, at least in a power series in the field theory coupling, the Vasiliev duals to the corresponding field theories are defined by an $f(X)$ that can be put in the form (5.30).

solves Vasiliev's equations. We refer to this solution as the AdS_4 vacuum (as we will see below this preserves the $SO(2,4)$ invariance of AdS space).

In the sequel we will find it convenient to work with a specific choice of coordinates and a specific choice of the vielbein field. For the metric on AdS space we work in Poincaré coordinates; the metric written in Euclidean signature takes the form

$$ds^2 = \frac{d\vec{x}^2 + dz^2}{z^2}, \tag{5.33}$$

We also define the vielbein one-form fields

$$e_0^i = -\frac{dx^i}{z}, \quad e_0^4 = -\frac{dz}{z} \tag{5.34}$$

(a runs over the index $i = 1 \ldots 3$ and $a = 4$). The corresponding spin connection one form fields are given by

$$w_0^{ab} = \frac{dx^i}{4z}\left[\mathrm{Tr}\left(\sigma^{iz}\sigma^{ab}\right) - \mathrm{Tr}\left(\bar{\sigma}^{iz}\bar{\sigma}^{ab}\right)\right] \tag{5.35}$$

Using (5.31) we have explicitly

$$\begin{aligned} \omega_0(x|Y) &= -\frac{1}{8}\frac{dx^i}{z}\left(y\sigma^{iz}y + \bar{y}\bar{\sigma}^{iz}\bar{y}\right), \\ e_0(x|Y) &= -\frac{1}{4}\frac{dx_\mu}{z}y\sigma^\mu\bar{y}. \end{aligned} \tag{5.36}$$

Here our convention for contracting spinor indices is $y\sigma^\mu\bar{y} = y^\alpha(\sigma^\mu)_\alpha{}^{\dot{\beta}}\bar{y}_{\dot{\beta}}$, etc (see Appendix 5.A.7).

Linearization around AdS

The linearization of Vasiliev's equations around the AdS solution of the previous subsection, yields Fronsdal's equations for the fields of all spins $s = 1, 2, \cdots, \infty$ together with the free minimally coupled equation for an $m^2 = -2$ scalar field. The demonstration of this fact is rather involved; we will not review it here but instead refer the reader to [22, 69] for details. In this subsubsection we content ourselves with reviewing a few structural features of linearized solutions that will be of use to us in the sequel.

In the linearization of Vasiliev's equations around AdS, it turns out that the the physical degrees of freedom are contained entirely in the master fields restricted to $Z \equiv (z_\alpha, \bar{z}_{\dot{\alpha}}) = 0$. The spin-$s$ degrees of freedom are contained in

$$\begin{aligned} \Omega^{(s-1+m,s-1-m)} &= W_\mu(x,Y,Z=0)|_{y^{s-1+m}\bar{y}^{s-1-m}}, \\ C^{(2s+n,n)} &= B(x,Y,Z=0)|_{y^{2s+n}\bar{y}^{n}}, \\ C^{(n,2s+n)} &= B(x,Y,Z=0)|_{y^{n}\bar{y}^{2s+n}}, \end{aligned} \tag{5.37}$$

for $-(s-1) \leq m \leq (s-1)$ and $n \geq 0$. In particular, $W(x,Y,Z=0)|_{y^{s-1}\bar{y}^{s-1}} = \Omega_{\alpha\dot{\beta}|\alpha_1\cdots\alpha_{s-1}\dot{\beta}_1\cdots\dot{\beta}_{s-1}} y^{\alpha_1}\cdots y^{A_{s-1}}\bar{y}^{\dot{\beta}_1}\cdots\bar{y}^{\dot{\beta}_{s-1}}dx^{\alpha\dot{\beta}}$ contains the rank-s symmetric (double-)traceless (metric-like) tensor gauge field[10], and $B|_{y^{2s}}, B|_{\bar{y}^{2s}}$ contain the self-dual and anti-self-dual parts of the higher spin generalization of the Weyl curvature tensor (and involve up to s spacetime derivatives on the symmetric tensor field). While the components of W_μ and B listed above are sufficient to recover all information about the spin s fields, they are not the only components of the Vasiliev field that are turned on in the linearized solution. The linearized Vasiliev equations relate the components

$$\begin{aligned} \cdots \leftarrow C^{(1,2s+1)} \leftarrow C^{(0,2s)} \leftarrow \Omega^{(0,2s-2)} \cdots \leftarrow \Omega^{(s-2,s)} \leftarrow \Omega^{(s-1,s-1)} \rightarrow \\ \hookrightarrow \Omega^{(s,s-2)} \rightarrow \cdots \Omega^{(2s-2,0)} \rightarrow C^{(2s,0)} \rightarrow C^{(2s+1,1)} \rightarrow \cdots \end{aligned} \tag{5.38}$$

Starting from $\Omega^{(s-1,s-1)}$, the arrows (to the left as well as to the right) are generated by the action of derivatives. This may schematically be understood as follows. $\Omega^{(s-1,s-1)}$ has $s-1$

[10] In order to formulate Fronsdal type equations with higher spin gauge symmetry of the form $\delta\varphi_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)} + \cdots$, the spin-$s$ gauge field is taken to be a rank-s symmetric double-traceless tensor field $\varphi_{\mu_1\cdots\mu_s}$. The trace part can be gauged away, however, leaving a symmetric rank-s traceless tensor.

symmetrized α type and $s-1$ symmetrized $\dot{\alpha}$ type indices. Acting with the derivative $\partial_{\gamma\dot{\beta}}$, symmetrizing γ with all the α type indices but contracting $\dot{\beta}$ with one of the $\dot{\alpha}$ type indices yields an object with s α type indices but only $s-2$ $\dot{\alpha}$ type indices, taking us along the right arrow from $\Omega^{(s-1,s-1)}$ in (5.38). A similar operation, interchanging the role of dotted and undotted indices takes us along to the left.

The equations for the metric-like fields $\varphi_{\mu_1\cdots\mu_s}$ of the standard form $(\Box - m^2)\varphi_{\mu_1\cdots\mu_s} + \cdots =$ (nonlinear terms) can be extracted from Vasiliev's equation by solving the auxiliary fields in terms of the metric-like fields order by order.

Parity

We wish to study Vasiliev's equations in an expansion around AdS space (with asymptotically AdS boundary conditions, as we will detail in the next section). Consider the action of a parity operation. In the coordinates of (5.33) this operation acts as $x^i \to -x^i$ (for $i = 1\ldots 3$). In order to fix the action of parity on the spinors y^α, $\bar{y}^{\dot{\alpha}}$ and z^α and $\bar{z}^{\dot{\alpha}}$ we adopt the choice of vielbein (5.34). With this choice the vielbeins are oriented along the coordinate axes and the parity operator on spinors takes the standard flat space form $\Gamma_5\Gamma_1\Gamma_2\Gamma_3 = \Gamma_4$. Using the explicit form for Γ_4 listed in (5.205), it follows that under parity

$$
\begin{aligned}
&\mathbf{P}(W(\vec{x}, z, d\vec{x}, dz|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = W(-\vec{x}, z, -d\vec{x}, dz|i(\sigma_z\bar{y})_\alpha, i(\sigma_z\bar{z})_\alpha, i(\sigma_z y)_{\dot{\alpha}}, i(\sigma_z z)_{\dot{\alpha}}),\\
&\mathbf{P}(S(\vec{x}, z|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = S(-\vec{x}, z|i(\sigma_z\bar{y})_\alpha, i(\sigma_z\bar{z})_\alpha, i(\sigma_z y)_{\dot{\alpha}}, i(\sigma_z z)_{\dot{\alpha}}),\\
&\mathbf{P}(B(\vec{x}, z|y_\alpha, z_\alpha, \bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})) = \pm B(-\vec{x}, z|i(\sigma_z\bar{y})_\alpha, i(\sigma_z\bar{z})_\alpha, i(\sigma_z y)_{\dot{\alpha}}, i(\sigma_z z)_{\dot{\alpha}})
\end{aligned}
\tag{5.39}
$$

(while the parity transformation of the one-form fields W and S are fixed by the transformations of dx^μ and dZ, the scalar B can be either parity odd or parity even). With the choice of conventions adapted in Appendix 5.A.7, $i\sigma_z = -I$. Consequently parity symmetry acts on (Y, Z) by exchanging $y_\alpha \leftrightarrow -\bar{y}_{\dot{\alpha}}$, $z_\alpha \leftrightarrow -\bar{z}_{\dot{\alpha}}$, and so exchanges the two terms $f_*(B*K)dz^2$ and $\overline{f}_*(B*\overline{K})d\bar{z}^2$ in the equation of motion.

When are Vasiliev's equations invariant under parity transformations? As we have seen above, B may be either parity even or odd. Thus we need either $f(X) = \overline{f}(X)$ or $f(X) = \overline{f}(-X)$. Combined with (5.30), we have

$$f_A(X) = \frac{1}{4} + X, \quad (\text{ A type}) \text{ or } f_B(X) = \frac{1}{4} + iX \quad (\text{ B type}) \tag{5.40}$$

They define the A-type and B-type Vasiliev theories, respectively.

Without imposing parity symmetry, however, the interactions of Vasiliev's system is governed by the function $f(X)$, or the phase $\theta(X)$. If $\theta(X)$ is not 0 or $\pi/2$, parity symmetry is violated. Parity symmetry is formally restored, however if we assign nontrivial parity transformation on $\theta(X)$ (i.e. on the coupling parameters θ_{2n}) as well; there are two ways of doing this, with the scalar master field B being parity even or odd:

$$\begin{aligned} &P_A: \quad B \to B, \quad \theta(X) \to -\theta(X), \quad \text{or} \\ &P_B: \quad B \to -B, \quad \theta(X) \to \pi - \theta(X). \end{aligned} \tag{5.41}$$

This will be useful in constraining the dependence of correlation functions on the coupling parameters θ_{2n}.

The duals of free theories

The bulk scalar of Vasiliev's theory turns out to have an effective mass $m^2 = -2$ in units of the AdS radius. Near the boundary $z = 0$ in the coordinates of (5.33) the equation of motion the bulk scalar field S to take the form

$$S \simeq az + bz^2 \tag{5.42}$$

while the bulk vector field takes the form

$$A_\mu \simeq a_\mu + j_\mu z \tag{5.43}$$

In order to completely specify Vasiliev's dynamical system we need to specify boundary conditions for the bulk scalar and vector fields (the unique consistent boundary condition of fields of higher spin is that they decay near the boundary like z^{s+1}).) We postpone a systematic study of boundary conditions to the next section. In this subsubsection we specify the boundary conditions that define, respectively, the Vasiliev dual to the theory of free bosons and free fermions.

The type A bosonic Vasiliev theory with $b = 0$ (for the unique bulk scalar) and $a_\mu = 0$ (for the unique bulk vector field) is conjectured to be dual to the theory of a single fundamental $U(N)$ boson coupled to $U(N)$ Chern-Simons theory at infinite level k. The primary single trace operators of this theory have quantum numbers

$$\sum_{s=0}^{\infty}(s+1, s)$$

(the first label above refers to the scaling dimension of the operator, while the second label its spin), exactly matching the linearized spectrum of type A Vasiliev theory. In Section 5.3.2 below we demonstrate that these are the only boundary conditions for the type A theory that preserve higher spin symmetry, the necessary and sufficient condition for these equations to be dual to the theory of free scalars [37].

The spectrum of primaries of a theory of free fermions subject to a $U(N)$ singlet condition is given by

$$(2, 0) + \sum_{s=1}^{\infty}(s+1, s)$$

This is exactly the spectrum of the type B Vasiliev theory with boundary conditions $a = a_\mu = 0$. It is not difficult to convince oneself that these are the unique boundary conditions for the type B theory that preserve conformal invariance; in Section 5.3.2 below that they also preserve the full the higher spin symmetry algebra, demonstrating that this Vasiliev system is dual to a theory of free fermions.

5.2.2 Nonabelian generalization

Vasiliev's system in AdS_4 admits an obvious generalization to non-abelian higher spin fields, through the introduction of Chan-Paton factors, much like in open string field theory. We simply replace the master fields W, S, B by $M \times M$ matrix valued fields, and replace the $*$-algebra in the gauge transformations and equations of motion by its tensor product with the algebra of $M \times M$ complex matrices. In making this generalization we modify neither the truncation (5.11) nor the reality condition (5.17) (except that the complex conjugation in (5.17) is now defined with Hermitian conjugation on the $M \times M$ matrices). We will refer to this system as Vasiliev's theory with $U(M)$ Chan-Paton factors.

One consequence of this replacement is that the $U(1)$ gauge field in the bulk turns into a $U(M)$ gauge field, and all other bulk fields are $M \times M$ matrices that transform in the adjoint of this gauge group.

It is natural to conjecture that the non-minimal bosonic Vasiliev theory with $U(M)$ Chan-Paton factors is then dual to $SU(N)$ vector model with M flavors. Take the example of A-type theory in AdS_4 with $\Delta = 1$ boundary condition. The dual CFT is that of NM free massless complex scalars ϕ_{ia}, $i = 1, \cdots, N$, $a = 1, \cdots M$, restricted to the $SU(N)$-singlet sector. The conserved higher spin currents are single trace operators in the adjoint of the

$U(M)$ global flavor symmetry. The dual bulk theory has a coupling constant $g \sim 1/\sqrt{N}$. The *bulk* 't Hooft coupling is then

$$\lambda = g^2 M \sim \frac{M}{N}. \tag{5.44}$$

We thus expect the bulk theory to be weakly coupled when $M/N \ll 1$. The latter will be referred to as the "vector model limit" of quiver type theories.

At the classical level the non abelian generalization of Vasiliev's theory has M^2 different massless spin s fields, and in particular M^2 different massless gravitons. This might appear to suggest that the dual field theory has M^2 exactly conserved stress tensors, in contradiction with general field theory lore for interacting field theories. In fact this is not the case. In Appendix 5.A.9 we argue that $\frac{1}{N}$ effects lift the scaling dimension of all but one of the M^2 apparent stress tensors for every choice of boundary conditions except the one that is dual to a theory of M^2 decoupled free scalar or fermionic boundary fields.

5.2.3 Supersymmetric extension

Following [58, 22, 59, 62, 63], to construct Vasiliev's system with extended supersymmetry, we introduce Grassmannian auxiliary variables ψ_i, $i = 1, \cdots, n$, that obey Clifford algebra $\{\psi_i, \psi_j\} = 2\delta_{ij}$, and commute with all the twistor variables (Y, Z). By definition, the ψ_i's do not participate in the $*$-algebra. The master fields W, S, B, as well as the gauge transformation parameter ϵ, are now all functions of ψ_i's as well as of $(x^\mu, y_\alpha, \bar{y}_{\dot\alpha}, z_\alpha, \bar{z}_{\dot\alpha})$.

The operators ψ_i may be thought of as Γ matrices that act on an auxiliary $2^{\frac{n}{2}}$ dimensional 'spinor' space (we assume from now on that n is even). Note that an arbitrary $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ dimensional matrix can be written as a linear sum of products of Γ matrices.[11] Consequently

[11]This fact gives a map from the space of $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ dimensional matrices to constant forms on an n

at this stage the extension of Vasiliev's system to allow for all fields to be functions of ψ_i is simply identical to the non abelian extension of the previous subsection, for the special case $M = 2^{\frac{n}{2}}$. The construction of this subsection differs from that of the previous one in the truncation we apply on fields. The condition (5.11) continues to take the form

$$[R, W]_* = \{R, S\}_* = [R, B]_* = [R, \epsilon]_* = 0. \tag{5.45}$$

but with R now defined as

$$R \equiv K\overline{K}\Gamma \tag{5.46}$$

and where

$$\Gamma \equiv i^{\frac{n(n-1)}{2}}\psi_1\psi_2\cdots\psi_n \tag{5.47}$$

(note that $\Gamma^2 = 1$ and that it is still true that $R * R = 1$).

While the modified truncation (5.45) looks formally similar to (5.11), it has one very important difference. As with (5.11) it ensures that those operators that commute with Γ (i.e. are even functions of ψ_i) are also even functions of the spinor variables Y, Z. However odd functions of ψ_i, which anticommute with Γ, are now forced to be odd functions of Y, Z. Such functions transform in spinorial representations of the internal tangent space $SO(4)$. Consequently, the new projection introduces bulk spinorial fields into Vasiliev's theory, and simultaneously ensures that such fields are always anticommuting, in agreement with the spin statistics theorem.

The reality projection we impose on fields is almost unchanged compared to (5.17). We

dimensional space, where ψ_i is regarded as a basis one-form. Every $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ dimensional matrix can be uniquely decomposed into the sum of a zero form $a_0 I$, a one form $a^i\psi_i$, a two form $a^{ij}\psi_i\psi_j$ … an n form $a_n\psi_1\psi_2...\psi_n$. The number of basis forms is $(1+1)^n = 2^n$, precisely matching the number of independent matrix elements.

demand

$$\iota(W)^* = -W, \quad \iota(S)^* = -S, \quad \iota(B)^* = \overline{K} * B * \overline{K}\Gamma = \Gamma K * B * K. \tag{5.48}$$

The operation ι and the complex conjugation on the master fields, $\mathcal{A} \to \mathcal{A}^*$, are defined in the Section 5.2.1, in combination with $\iota: \ \psi_i \to \psi_i$ but reverses the order of the product of ψ_i's, and ψ_i's are real under complex conjugation. We require ι to reverse the order of ψ_i in order to ensure that

$$\iota(\Gamma)^* = \Gamma^{-1} = \Gamma.$$

(the reversal in the order of ψ_i compensates for the sign picked up by the factor of $i^{\frac{n(n-1)}{2}}$ under complex conjugation in (5.47)). The only other modification in (5.48) compared to (5.17) is in the factor on Γ in the action on B; this additional factor is necessary in order for the two terms on the RHS of $\iota(B)^*$ to be the same, after using the truncation equation (5.45), given that R in this section has an additional factor of Γ as compared to the bosonic theory.

Vasiliev's equations take the form

$$\begin{aligned} &\mathcal{F} = d_x\hat{\mathcal{A}} + \hat{\mathcal{A}} * \hat{\mathcal{A}} = f_*(B * K)dz^2 + \overline{f}_*(B * \overline{K}\Gamma)d\bar{z}^2, \\ &d_xB + \hat{\mathcal{A}} * B - B * \pi(\hat{\mathcal{A}}) = 0. \end{aligned} \tag{5.49}$$

Compared to the bosonic theory, the only change in the first Vasiliev equation is the factor of Γ in the argument of $\overline{f}$; this factor is needed in order to preserve the reality of Vasiliev equations under the operation (5.48), as it follows from (5.48) that

$$\iota(B * K)^* = \overline{K} * \overline{K} * B * \overline{K}\Gamma = B * \overline{K}\Gamma.$$

The second Vasiliev equation is unchanged in form from the bosonic theory; however the operator π is now taken to mean conjugation by $\Gamma\overline{K}$ together with $d\bar{z} \to -d\bar{z}$, or equivalently, by the truncation condition (5.45) on the fields, conjugation by K together with $dz \to -dz$. Note in particular that

$$\begin{aligned}\pi(S) &= K * S_{\overline{z}} * K + \Gamma\overline{K} * S_z * \Gamma\overline{K} \\ &= S_{\dot{\alpha}}(x|-y,\bar{y},-z,\bar{z},\psi)d\bar{z}^{\dot{\alpha}} + S_{\alpha}(x|y,-\bar{y},z,-\bar{z},-\psi)dz^{\alpha} \\ &= S(x|y,-\bar{y},z,-\bar{z},-\psi,dz,-d\bar{z}).\end{aligned} \tag{5.50}$$

As in the case of the bosonic theory, $f(X)$ can generically be cast into the form $f(X) = \frac{1}{4} + X\exp(i\theta(X))$ by a field redefinition.

The expansion into components of the first of (5.49) is given by (5.193), with the last line of that equation replaced by

$$\hat{S} * \hat{S} = f(B * K)dz^2 + \bar{f}(B * \overline{K}\Gamma)\bar{z}^2, \tag{5.51}$$

The expansion in components of the second line of (5.49) is given by (5.194) with no modifications.

As in the case of the bosonic theory, the second equation in (5.49) follows from the first using the Bianchi identity for the field strength. The details of the derivation differ in only minor ways from the bosonic derivation presented in Appendix 5.A.4.[12]

Parity acts as

$$\begin{aligned}&\mathbf{P}(W(\vec{x},z,d\vec{x},dz|y_\alpha,z_\alpha,\bar{y}_{\dot{\alpha}},\bar{z}_{\dot{\alpha}})) = W(-\vec{x},z,-d\vec{x},dz|i(\sigma_z\bar{y})_\alpha,i(\sigma_z\bar{z})_\alpha,i(\sigma_z y)_{\dot{\alpha}},i(\sigma_z z)_{\dot{\alpha}}),\\ &\mathbf{P}(S(\vec{x},z|y_\alpha,z_\alpha,\bar{y}_{\dot{\alpha}},\bar{z}_{\dot{\alpha}})) = S(-\vec{x},z|i(\sigma_z\bar{y})_\alpha,i(\sigma_z\bar{z})_\alpha,i(\sigma_z y)_{\dot{\alpha}},i(\sigma_z z)_{\dot{\alpha}}),\\ &\mathbf{P}(B(\vec{x},z|y_\alpha,z_\alpha,\bar{y}_{\dot{\alpha}},\bar{z}_{\dot{\alpha}})) = B(-\vec{x},z|i(\sigma_z\bar{y})_\alpha,i(\sigma_z\bar{z})_\alpha,i(\sigma_z y)_{\dot{\alpha}},i(\sigma_z z)_{\dot{\alpha}})\,\Gamma.\end{aligned} \tag{5.52}$$

[12](5.196) holds unchanged, (5.197) holds with $\overline{K} \to \overline{K}\Gamma$ these two equations are equivalent by (5.45). Equation (5.199) holds unchanged. (5.201) applies with $\overline{K} \to \overline{K}\Gamma$. (5.202) holds unchanged.

The factor of Γ in the last of (5.52) is needed in order that the theory with $f(X) = \frac{1}{4} + X$ is parity invariant.

5.2.4 The free dual of the parity preserving susy theory

In this subsection we consider the dual description of the parity preserving Vasiliev theory with appropriate boundary conditions. The equations we study have $f(X) = \frac{1}{4} + X$. Let us examine the bulk scalar fields which are given by the bottom component of the B master field, namely $\Phi(x, \psi) = B(x|Y = Z = 0, \psi)$, which obeys the truncation condition $\Gamma\Phi\Gamma = \Phi$, i.e. Φ is even in the ψ_i's. There are 2^{n-1} real scalars, half of which are parity even, the other half parity odd. We impose boundary conditions to ensure that $\Delta = 1$ for the parity even scalars and $\Delta = 2$ for the parity odd scalars (see the next sections for details). In other words the fall off near the boundary is given by (5.42), with $b = 0$ for parity even scalars, $a = 0$ for all parity odd scalars. The boundary fall off for all gauge fields is given by (5.43) with $a_\mu = 0$.

The bulk theory has also $m = 0$ spin half bulk fermions, whose boundary conditions we now specify. Recall (see e.g. [70]) that the AdS/CFT dictionary for such fermions identifies the 'source' with the coefficient of the $z^{\frac{3}{2}}$ fall off of the parity even part of the bulk fermionic field (the same information is also present in the $z^{\frac{5}{2}}$ fall off of the parity odd part of the fermion field), while the 'operator vev' is identified with the coefficient of the $z^{\frac{3}{2}}$ of the parity odd part of the bulk fermion field (the same information is also present in the $z^{\frac{5}{2}}$ fall off of the parity even part of the fermion field). We impose the standard boundary conditions that set all sources to zero, i.e. we demand that the leading $\mathcal{O}(z^{\frac{3}{2}})$ fall off of the fermionic field is entirely parity odd. We believe these boundary conditions preserve the fermionic higher spin symmetry (see Section 5.5.4 for a partial verification) and so yield the theory dual to a free field theory.

The field content of this dual field theory is as follows; we have $2^{\frac{n}{2}-1}$ complex scalars in the fundamental representation and the same number of fundamental fermions (so that the singlets constructed out of bilinears of scalars or fermions match with the bulk scalars). We organize the fields in the boundary theory in the form

$$\phi_{iA}, \quad \psi_{i\dot{B}\alpha},$$

where i is the $SU(N)$ index, $A, \dot{B}$ are chiral and anti-chiral spinor indices of an $SO(n)$ global symmetry, and α denotes the spacetime spinor index of $\psi_{i\dot{B}}$. The $2^{n-2}+2^{n-2}$ $SU(N)$ singlet scalar operators, of dimension $\Delta=1$ and $\Delta=2$, are

$$\bar{\phi}^{iA}\phi_{iB}, \quad \bar{\psi}^{i\dot{A}}\psi_{i\dot{B}}. \tag{5.53}$$

They are dual to the bulk fields (projected to the parity even and parity odd components, respectively)

$$\Phi_{+} = \Phi\frac{1+\Gamma}{2}, \quad \Phi_{-} = -i\Phi\frac{1-\Gamma}{2}. \tag{5.54}$$

The free CFT has $U(2^{\frac{n}{2}-1})\times U(2^{\frac{n}{2}-1})$ bosonic flavor symmetry that act on the scalars and fermions separately, as well as 2^{n-2} complex fermionic symmetry currents

$$(J_{\alpha\mu})^{\dot{B}}{}_{A} = \bar{\psi}^{i\dot{B}}_{\alpha}\overleftrightarrow{\partial}_{\mu}\phi_{iA} + \cdots . \tag{5.55}$$

The Vasiliev bulk dual of the $U(2^{\frac{n}{2}-1})\times U(2^{\frac{n}{2}-1})$ global symmetry is given by Vasiliev gauge transformations with ϵ independent of x, Y or Z, but an arbitrary real even function of ψ_i (i.e. an arbitrary even Hermitian operator built out of ψ_i). Operators of this nature may be subdivided into parity even and parity odd Hermitian operators which mutually commute. The 2^{n-2} parity even operators of this nature generate one factor of $U(2^{\frac{n}{2}-1})$ while the complementary parity even operators generate the second factor. The two central $U(1)$ elements are generated by $I+\Gamma$ and $I-\Gamma$ respectively; these operators clearly commute with all even functions of ψ_i, and so commute with all other generators, establishing their central nature.[13] It is easily verified that parity even Vasiliev scalars transform are neutral under the parity odd $U(2^{\frac{n}{2}-1})$ but transform in the adjoint of the parity even $U(2^{\frac{n}{2}-1})$ (the reverse statement is also true). On the other hand the parity even/odd spin half fields of Vasiliev theory transform in the (fundamental, antifundamental) and (fundamental, antifundamental), all in agreement with field theory expectations.

With the boundary conditions described in this section, the bulk theory may be equivalently written as the $n'=2$ (i.e. minimally) extended supersymmetric Vasiliev theory with $U(2^{\frac{n}{2}-1})$ Chan-Paton factors and boundary conditions that preserved this symmetry. Our main interest in the bulk dual of the free theory, however, is as the starting point for the construction of the bulk dual of interacting theories. This will necessitate the introduction of parity violating phases into the theory and simultaneously modifying boundary conditions. The boundary conditions we will introduce break the $U(2^{\frac{n}{2}-1})$ global symmetry down to a smaller subgroup. In every case of interest the subgroup in question will turn out to be a subgroup of $U(2^{\frac{n}{2}-1})$ that is also a subgroup of the $SO(n)$[14] that rotates the ψ_i's (here ψ_i are

[13]As an example let us consider the case $n=4$ that is of particular interest to us below. The parity even $U(2)=U(1)\times SU(2)$ is generated by

$$(1+\Gamma), \quad (1+\Gamma)\psi_4\psi_i$$

while the parity odd $U(2)=U(1)\times SU(2)$ is generated by

$$(1-\Gamma), \quad (1-\Gamma)\psi_4\psi_i$$

(where $i=1\ldots 3$).

[14]As we will see in the sequel, we will find it possible to choose boundary conditions to preserve up to $\mathcal{N}=6$ supersymmetries together with a flavour symmetry group which is a subgroup of $U(2^{\frac{n}{2}-1})\times U(2^{\frac{n}{2}-1})$.

the fermionic fields that enter Vasiliev's construction, not the fermions of the dual boundary theory). As the preserved symmetry algebras have a natural action on ψ_i, the language of extended supersymmetry will prove considerably more useful for us in subsequent sections than the language of the non abelian extension of the $n = 2$ theory, which we will never adopt in the rest of this paper.

5.3 Higher Spin symmetry breaking by AdS$_4$ boundary conditions

In this technical section, we will demonstrate that higher spin bulk symmetries are broken by nontrivial values of the phase function θ and by generic boundary conditions.

In this section we study mainly the bosonic Vasiliev theory. We demonstrate that higher spin symmetry is broken by generic boundary conditions and generic values of the Vasiliev phase. Higher spin symmetry is preserved *only* for the type A and type B Vasiliev theories with boundary conditions described in Section 5.2.1. We will see this explicitly by showing that, in every other case, the *nonlinear* (higher) spin-s gauge transformation on the bulk scalar field, at the presence of a spin-s' boundary source, violates the boundary condition for the scalar field itself for every other choice of phase or boundary condition. We also use this bulk analysis together with a Ward identity to compute the coefficient $c_{ss'0}$ in the schematic equation

$$\partial^\mu J^{(s)}_\mu = c_{ss'0} J^{s'} O + \cdots$$

where the RHS includes the contributions of descendants of $J^{s'}$ and descendants of O. The violation of the scalar boundary condition is directly related to a double trace term in the anomalous "conservation" law of the boundary spin-s current, via a Ward identity.

This section does not directly relate to the study of the bulk duals of supersymmetric Chern Simons theories. Apart from the basic formalism for the study of symmetries in Vasiliev theory (see Section 5.3.1 below) the only result of this subsection that we will use later in the paper are the identifications (5.83) and (5.86) presented below. The reader who is willing to take these results on faith, and who is uninterested in the bulk mechanism of higher spin symmetry breaking, could skip directly from Section (5.3.1) to the next section.

5.3.1 Symmetries that preserve the AdS Solution

The asymptotic symmetry group of Vasiliev theory in AdS_4 is generated by gauge parameters $\epsilon(x|Y, Z, \psi_i)$ that leave the AdS_4 vacuum solution (5.32) invariant. $S = 0$ in the solution (5.32) is preserved if and only if the gauge transformation parameter is independent of Z, i.e it takes the form $\epsilon(x|Y, \psi_i)$. As B transforms homogeneously under gauge transformations, $B = 0$ (in (5.32)) is preserved under arbitrary gauge transformations. The nontrivial conditions on $\epsilon(x|Y, \psi_i)$ arise from requiring that $W = W_0$ is preserved. For this to be the case $\epsilon(x|Y, \psi_i)$ is required to obey the equation

$$D_0\epsilon(x|Y, \psi_i) \equiv d_x\epsilon(x|Y, \psi_i) + [W_0, \epsilon(x|Y, \psi_i)]_* = 0. \tag{5.56}$$

As the gauge field W_0 in the AdS_4 vacuum obeys the equation $d_xW_0 + W_0 * W_0 = 0$, W_0 is a flat connection and so may may be written in the "pure gauge" form.

$$W_0 = L^{-1} * dL, \tag{5.57}$$

where L^{-1} is the $*$-inverse of $L(x|Y)$. We may formally move to the gauge in which $W_0 = 0$;[15]

[15]Note that the formal gauge transformation by L is not a true gauge symmetry of the theory, as it violates the AdS boundary condition. We regard it as merely a solution generating technique.

$W = 0$ is preserved if and only if ϵ is independent of x. Transforming back to the original gauge we conclude that the most general solution to (5.56) is given by $\epsilon(x|Y)$ of the form

$$\epsilon(x|Y, \psi_i) = L^{-1}(x|Y) * \epsilon_0(Y, \psi_i) * L(x|Y). \tag{5.58}$$

where $\epsilon_0(Y)$ is independent of x and is restricted, by the truncation condition, to be an even function of y, ψ_i.[16]

The gauge function $L(x|Y)$ is not uniquely defined; it may be obtained by integrating the flat connection W_0 along a path from a base point x_0 to x. We would then have $L(x_0|Y) = 1$ and $\epsilon_0(Y) = \epsilon(x_0|Y)$. See [71, 33] for explicit formulae for $L(x|Y)$ in Poincaré coordinates. We have used the explicit form of $L(x|Y)$ to obtain an explicit form for $\epsilon(x|Y)$. We now describe our final result, which may easily independently be verified to obey (5.56)

Let us define $y_\pm \equiv y \pm \sigma^z \bar{y}$. The $*$-contraction between $y_\pm$ and $y_\pm$ is zero, and is nonzero only between $y_\pm$ and $y_\mp$. Namely, we have

$$\begin{aligned} (y_\pm)_\alpha * (y_\pm)_\beta &= (y_\pm)_\alpha (y_\pm)_\beta, \\ (y_\pm)_\alpha * (y_\mp)_\beta &= (y_\pm)_\alpha (y_\mp)_\beta + 2\epsilon_{\alpha\beta}. \end{aligned} \tag{5.59}$$

In Poincaré coordinates, W_0 may be written in terms of $y_\pm$ as

$$W_0 = -\frac{dx^i}{8z} y_+ \sigma^{iz} y_+ + \frac{dz}{8z} y_+ y_-. \tag{5.60}$$

A generating function for solutions to (5.56) is given by

$$\begin{aligned} \epsilon(x|Y) &= \exp\left[z^{-\frac{1}{2}} \Lambda_+(\vec{x}) y_+ + z^{\frac{1}{2}} \Lambda_- y_- \right] \\ &= \exp\left[\Lambda(x) y + \overline{\Lambda}(x) \bar{y} \right], \end{aligned} \tag{5.61}$$

[16] This is obvious in the gauge in which W vanishes. In the gauge (5.58) it follows from the truncation condition $[\epsilon, R]_* = 0$, and that the fact that $[L(x|Y), R]_* = 0$, we see that $\epsilon_0(Y)$.

where $\Lambda_+(\vec{x}), \Lambda(\vec{x})$, and $\overline{\Lambda}(\vec{x})$ are given in terms of constant spinors Λ_0 and Λ_- by

$$\begin{aligned}
\Lambda_+(\vec{x}) &= \Lambda_0 + \vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-, \\
\Lambda(x) &= z^{-\frac{1}{2}}\Lambda_+(\vec{x}) + z^{\frac{1}{2}}\Lambda_-, \\
\overline{\Lambda}(x) &= -z^{-\frac{1}{2}}\sigma^z\Lambda_+(\vec{x}) + z^{\frac{1}{2}}\sigma^z\Lambda_-.
\end{aligned} \tag{5.62}$$

$\epsilon(x|Y)$ as defined in (5.61) may directly be verified to obey the linear equation (5.56). (5.61) is a generating function for solutions to that equation in the usual: upon expanding $\epsilon(x|Y)$ in a power series in the arbitrary constant spinors Λ_0 and Λ_- the coefficients of different powers in this Taylor expansion independently obey (5.56) (this follows immediately from the linearity of (5.56)).

Notice that the various Taylor coefficients in (5.61) contains precisely all generating parameters for the universal enveloping algebra of $so(3,2)$ (in the bosonic case) or its appropriate supersymmetric extension (in the susy case).

Let us first describe the bosonic case. Recall that, on the boundary, the conserved currents of the higher spin algebra may be obtained by dotting a spin s conserved current with $s-1$ conformal killing vectors. Let us define the 'spin s charges' as the charges obtained out of the spin s conserved current by this dotting process. The spin-s global symmetry generating parameter, $\epsilon^{(s)}(x|Y)$, is then obtained from the terms in (5.61) of homogeneous degree $2s-2$ in $(y,\bar{y})$ (or equivalently in Λ_0 and Λ_-).

As a special case consider the 'spin two' charges, i.e. the charges whose conserved currents correspond to the stress tensor dotted with a single conformal killing vector, i.e the conformal generators. These generators are quadratic in $(y,\bar{y})$. These generators may be organized under the action of the boundary $SU(2)$ (i.e. the diagonal action of $SU(2)_L$ and $SU(2)_R$) as $3+3+3+1$, corresponding to 3d angular momentum generators, momenta, boosts and dilations, in perfect correspondence with generators of the three dimensional conformal group $so(3,2)$.[17] Indeed the set of quadratic Hamiltonians in Y, with product defined by the star algebra, provides an oscillator construction of $so(3,2)$.

Let us now turn to the supersymmetric theory. The generators of the full n extended superconformal algebra are given by terms that are quadratic in (y, ψ_i). Terms quadratic in y are conformal generators. Terms quadratic in ψ_i but independent of y are $SO(n)$ R symmetry generators. Terms linear in both y and ψ_i (we denote these by $\epsilon^{(\frac{3}{2})}(x|Y)$) are supersymmetry and superconformal generators. More precisely the terms involving Λ_0 are Poincaré supersymmetry parameters, where the terms involving Λ_- are special supersymmetry generators (in radial quantization with respect to the origin $\vec{x} = 0$).

In the sequel we will will make use of the following easily verified algebraic property of the generating function $\epsilon(x|Y)$ (5.61) under $*$ product,

$$\begin{aligned} \epsilon(x|Y) * f(y, \bar{y}) &= \epsilon(x|Y) f(y + \Lambda, \bar{y} + \overline{\Lambda}), \\ f(y, \bar{y}) * \epsilon(x|Y) &= \epsilon(x|Y) f(y - \Lambda, \bar{y} - \overline{\Lambda}). \end{aligned} \tag{5.63}$$

5.3.2 Breaking of higher spin symmetries by boundary conditions

Any given Vasiliev theory is defined by its equations of motion together with boundary conditions for all fields. Given any particular boundary conditions one may ask the following question: which of the large gauge transformation described in the previous subsection preserve these boundary conditions? In other words which if any of the gauge transformations have the property that they return a normalizable state (i.e. a solution of Vasiliev's theory that obeys the prescribed boundary conditions) when acting on an arbitrary normalizable

[17]It may be checked that The Poincaré generators are obtained by simply setting Λ_- to zero.

state? Such gauge transformations are genuine global symmetries of the system.

In this paper we will study the exact action of the large gauge transformations of the previous section on an arbitrary *linearized* solution of Vasiliev's equations. The most general such solution may be obtained by superposition of the linearized responses to arbitrary boundary sources. Because of the linearity of the problem, it is adequate to study these sources one at a time. Consequently we focus on the linearized solution created by a spin s source at $x = 0$ on the boundary of AdS_4. Such a source creates a response of the B field everywhere in AdS_4, and in particular in the neighborhood of the boundary at the point x. We study the higher spin gauge transformations $\epsilon^{(s')}(x|Y)$ (for arbitrary s') on the B master field at this point. The response to this gauge variation contains fields of various spins s''. As we will see below the response for $s'' > 1$ always respects the standard boundary conditions for spin s'' fields. However the same is not true of the response of the fields of low spins, namely $s'' = 0, \frac{1}{2}$, or 1. As we have seen in the previous section, for these fields it is possible to choose different boundary conditions, some of which turn out to be violated by the symmetry variation δB.

In the rest of this section we restrict our attention to the bosonic Vasiliev theory. The variation δB under an asymptotic symmetry generated by $\epsilon(x|Y)$ in (5.61) is given by (5.7).

Let $B^{(s)}(x|Y)$ be the spin-s component of the linearized $B(x|Y)$ sourced by a current $J^{(s)}$ on the boundary, i.e. the boundary to bulk propagator for the spin-s component of the B master field with the source inserted at $\vec{x} = 0$. $B^{(s)}(x|Y)$ only contains terms of order $y^{2s+n}\bar{y}^n$ and $y^n\bar{y}^{2s+n}$, $n \geq 0$; as we have explained above, the coefficients of these terms are spacetime derivatives of the basic spin s field. We will work in Poincaré coordinates (5.33), with the spin-s source located at $\vec{x} = 0$. Without loss of generality, it suffices to consider the polarization tensor for $J^{(s)}$, a three-dimensional symmetric traceless rank-s tensor, of the form $\varepsilon_{\alpha_1\cdots\alpha_{2s}} = \lambda_{\alpha_1}\cdots\lambda_{\alpha_{2s}}$, for an arbitrary polarization spinor λ. The corresponding boundary-to-bulk propagator is computed in [33]. Here we generalize it slightly to the parity violating theory, by including the interaction phase $e^{i\theta_0}$, as

$$B^{(s)}(x|Y) = \frac{z^{s+1}}{(\vec{x}^2+z^2)^{2s+1}} e^{-y\Sigma\bar{y}} \left[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z y)^{2s} + e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^{2s}\right], \tag{5.64}$$

where Σ and $\mathbf{x}$ are defined as

$$\Sigma \equiv \sigma^z - \frac{2z}{\vec{x}^2+z^2}\mathbf{x}, \qquad \mathbf{x} \equiv x^\mu\sigma_\mu = \vec{x}\cdot\vec{\sigma} + z\sigma^z. \tag{5.65}$$

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Note that this formula is valid for spin $s > 1$, for the standard "magnetic" boundary condition in the $s = 1$ case and for $\Delta = 1$ boundary condition in the $s = 0$ case. The variation of B under the asymptotic symmetry generated by $\epsilon(x|Y)$ is given by

$$\begin{aligned}\delta B &= -\epsilon * B^{(s)} + B^{(s)} * \pi(\epsilon)\\ &= -\epsilon(x|y,\bar{y})B(x|y+\Lambda,\bar{y}+\overline{\Lambda}) + \epsilon(x|y,-\bar{y})B(x|y-\Lambda,\bar{y}+\overline{\Lambda}),\end{aligned}$$

where we made use of the properties (5.63). Using the explicit expression of the boundary-

[18]In the special case $s = 0$ the terms in the square bracket reduce simply to $2\cos\theta_0$. This observation is presumably related to the fact, discussed by Maldacena and Zibhoedov [51], that the scalar and spin s currents in the higher spin multiplets have different natural normalizations. In the sequel we will, indeed, identify the factor of $\cos\theta_0$ with the ratio of these normalizations.

to-bulk propagator, this is

$$
\begin{aligned}
\delta B = & -\frac{z^{s+1}}{(\vec{x}^2+z^2)^{2s+1}}\Big\{ e^{\Lambda y+\overline{\Lambda}\bar{y}} e^{-(y+\Lambda)\Sigma(\bar{y}+\overline{\Lambda})}\left[e^{i\theta_0}(\lambda \mathbf{x}\sigma^z(y+\Lambda))^{2s}+e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z(\bar{y}+\overline{\Lambda}))^{2s}\right] \\
& \quad - e^{\Lambda y-\overline{\Lambda}\bar{y}} e^{-(y-\Lambda)\Sigma(\bar{y}+\overline{\Lambda})}\left[e^{i\theta_0}(\lambda \mathbf{x}\sigma^z(y-\Lambda))^{2s}+e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z(\bar{y}+\overline{\Lambda}))^{2s}\right]\Big\} \\
= & -\frac{z^{s+1}}{(\vec{x}^2+z^2)^{2s+1}} e^{-y\Sigma\bar{y}+z^{-\frac12}\Lambda_+(1-\sigma_z\Sigma)y+z^{\frac12}\Lambda_-(1+\sigma_z\Sigma)y} \\
& \times\Big\{ e^{(z^{-\frac12}\Lambda_++z^{\frac12}\Lambda_-)\Sigma\sigma^z(z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-)+z^{-\frac12}\Lambda_+(\sigma^z-\Sigma)\bar{y}-z^{\frac12}\Lambda_-(\sigma^z+\Sigma)\bar{y}} \\
& \quad \times\left[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z(y+z^{-\frac12}\Lambda_++z^{\frac12}\Lambda_-))^{2s}+e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z(\bar{y}-\sigma^z(z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-)))^{2s}\right] \\
& -e^{-(z^{-\frac12}\Lambda_++z^{\frac12}\Lambda_-)\Sigma\sigma^z(z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-)-z^{-\frac12}\Lambda_+(\sigma^z-\Sigma)\bar{y}+z^{\frac12}\Lambda_-(\sigma^z+\Sigma)\bar{y}} \\
& \quad \times\left[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z(y-z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-))^{2s}+e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\sigma^z(\bar{y}-\sigma^z(z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-)))^{2s}\right]\Big\}.
\end{aligned} \tag{5.66}
$$

Note that although the source is a spin-s current, there are nonzero variation of fields of various spins in δB. The self-dual part of the higher spin Weyl tensor, in particular, is obtained by restricting $B(x|Y)$ to $\bar{y}=0$. The variation of the self-dual part of the Weyl tensors of various spins are given by

$$
\begin{aligned}
\delta B|_{\bar{y}=0} = & -\frac{z^{s+1}}{(\vec{x}^2+z^2)^{2s+1}} e^{z^{-\frac12}\Lambda_+(1-\sigma_z\Sigma)y+z^{\frac12}\Lambda_-(1+\sigma_z\Sigma)y} \\
\times & \Big\{ e^{(z^{-\frac12}\Lambda_++z^{\frac12}\Lambda_-)\Sigma\sigma^z(z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-)}\left[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z(y+z^{-\frac12}\Lambda_++z^{\frac12}\Lambda_-))^{2s}+e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}(z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-))^{2s}\right] \\
- & e^{-(z^{-\frac12}\Lambda_++z^{\frac12}\Lambda_-)\Sigma\sigma^z(z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-)}\left[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z(y-z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-))^{2s}+e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}(z^{-\frac12}\Lambda_+-z^{\frac12}\Lambda_-))^{2s}\right]\Big\}.
\end{aligned} \tag{5.67}
$$

Now let us examine the behavior of δB near the boundary of AdS$_4$. In the $z\to 0$ limit, the leading order terms in z are given by

$$
\begin{aligned}
\delta B|_{\bar{y}=0} \quad \longrightarrow \quad & -\frac{z}{|x|^{4s+2}} e^{2z^{\frac12}\left(\frac{1}{|x|^2}\Lambda_+\sigma^z\mathbf{x}+\Lambda_-\right)y} \\
& \times\Big\{ e^{\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+-2\Lambda_+\Lambda_-}\left[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z(z^{\frac12}y+\Lambda_+))^{2s}+e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\Lambda_+)^{2s}\right] \\
& \quad - e^{-\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_++2\Lambda_+\Lambda_-}\left[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z(z^{\frac12}y-\Lambda_+))^{2s}+e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\Lambda_+)^{2s}\right]\Big\}
\end{aligned} \tag{5.68}
$$

The variation of the spin-s'' Weyl tensor, $\delta B^{(s'')}$, is extracted from terms of order $y^{2s''}$ in the above formula, which falls off like $z^{s''+1}$ as $z \to 0$. This is consistent with the boundary condition for fields of spin $s'' > 1$, independently of the phase θ_0. As promised above, the spin $s'' > 1$ component of the response to an arbitrary gauge variation *automatically* obeys the prescribed boundary conditions for such field and so appears to yield no restrictions on allowed boundary conditions for the theory.

Anomalous higher spin symmetry variation of the scalar

The main difference between the scalar field and fields of arbitrary spin is that the prescribed boundary conditions for scalars involve both the leading as well as the subleading fall off of the scalar field. So while the leading fall off of the scalar field will never be faster than z^1 (in agreement with the general analysis above upon setting $s'' = 0$), this is not sufficient to ensure that the scalar field variation obeys its boundary conditions.

Let us examine the variation of the scalar field due to a higher spin gauge transformation, at the presence of a spin-s source at $\vec{x} = 0$ on the boundary. The spin $s'' = 0$ component of the symmetry variation δB is given by (5.67) with $(y, \bar{y})$ set to zero,

$$\begin{aligned}
\delta B^{(0)} &= -2\frac{z}{(\vec{x}^2+z^2)^{2s+1}} \sinh\left[(z^{-\frac{1}{2}}\Lambda_+ + z^{\frac{1}{2}}\Lambda_-)\Sigma\sigma^z(z^{-\frac{1}{2}}\Lambda_+ - z^{\frac{1}{2}}\Lambda_-)\right] \\
&\quad \times \left[e^{i\theta_0}(\lambda \mathbf{x}\sigma^z(\Lambda_+ + z\Lambda_-))^{2s} + e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}(\Lambda_+ - z\Lambda_-))^{2s}\right] \\
&= \frac{4}{(\vec{x}^2+z^2)^{2s+1}} \sinh\left[2\frac{\vec{x}^2-z^2}{\vec{x}^2+z^2}(\Lambda_+\Lambda_-) + 2\frac{\Lambda_+\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+ - z^2\Lambda_-\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-}{\vec{x}^2+z^2}\right] \\
&\quad \times \left[\cos\theta_0(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)^{2s}z + i\sin\theta_0\cdot 2s(\lambda(\Lambda_+ + \vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-))(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)^{2s-1}z^2 + \mathcal{O}(z^3)\right].
\end{aligned} \tag{5.69}$$

When expanded in a power series in Λ, the RHS of (5.69) has the schematic form

$$\mathcal{O}(\Lambda^{2s+2}) \times \text{ (Taylor expansion in } \Lambda^4)$$

Recall that the spin-s' symmetry variation (see the previous subsection for a definition) is extracted from terms of order $2s'-2$ in $\Lambda_\pm$. It follows that we find a scalar response to spin s' gauge transformations only for $s' = s+2, s+4, \ldots$. When this is the case (i.e. when $s'-s$ is positive and even)

$$\delta_{(s')}B^{(0)} = \frac{4}{(\vec{x}^2)^{2s+1}}\frac{2^{s'-s-1}}{(s'-s-1)!}\left(\Lambda_+\Lambda_- + \frac{1}{\vec{x}^2}\Lambda_+\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+\right)^{s'-s-1}$$
$$\times\left[\cos\theta_0(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)^{2s}z + i\sin\theta_0\cdot 2s(\lambda(\Lambda_+ + \vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-))(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)^{2s-1}z^2 + \mathcal{O}(z^3)\right]. \tag{5.70}$$

Recall that $\Lambda_+ = \Lambda_0 + \vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-$, and Λ_0, Λ_- are arbitrary constant spinors. For generic parity violating phase θ_0, and $s' > s > 0$ with even $s'-s$, terms of order z and z^2 are both nonzero, and so both $\Delta = 1$ and $\Delta = 2$ boundary conditions would be violated, leading to the breaking of spin-s' symmetry.

Note that the condition $s' > s > 0$ and that $s'-s$ is even means that the broken symmetry has spin $s' > 2$. In particular the $s' = 2$ conformal symmetries are never broken.[19]

The exceptional cases are when either $\cos\theta_0 = 0$ or $\sin\theta_0 = 0$. These are precisely the interaction phase of the parity invariant theories. In the A-type theory, $\theta_0 = 0$, we see that $\delta B^{(0,0)} \sim z + \mathcal{O}(z^3)$, and so $\Delta = 1$ boundary condition is preserved while $\Delta = 2$ boundary condition would be violated. This is as expected: the A-type theory with $\Delta = 1$ boundary condition is dual to the free $U(N)$ or $O(N)$ theory which has exact higher spin symmetry, whereas the A-type theory with $\Delta = 2$ boundary condition is dual to the critical theory, where the higher spin symmetry is broken at order $1/N$. For the B-type theory, $\theta_0 = \pi/2$, we see that $\delta B^{(0,0)} \sim z^2 + \mathcal{O}(z^3)$, and so the $\Delta = 2$ boundary condition is preserved, while $\Delta = 1$ boundary condition is violated. This is in agreement with the former case being dual

[19]Note that the extrapolation of this formula to the $s = 0$ case assumes $\Delta = 1$ boundary to bulk propagator, and the variation $\delta_{(s')}B^{(0)}$ is always consistent with the $\Delta = 1$ boundary condition.

to free fermions, and the latter dual to critical Gross-Neveu model where the higher spin symmetry is broken.

In summary, the *only* conditions under which *any* higher spin symmetries are preserved are the type A theory with $\Delta = 1$ or the type B theory with $\Delta = 2$. These are precisely the theories conjectured to be dual to the free boson and free fermion theory respectively, in agreement with the results of [37].

Ward identity and current non-conservation relation

To quantify the breaking of higher spin symmetry, we now derive a sort of Ward identity that relates the anomalous spin-s symmetry variation of the bulk fields, as seen above, to the non-conservation relation of the three-dimensional spin-s' current that generates the corresponding global symmetry of the boundary CFT.

Let us first word the argument in boundary field theory language. Let us consider the field theory quantity

$$\langle J^s(0) \cdots \rangle$$

where ... denote arbitrary current insertions away from the point x^μ, and $\langle\ \rangle$ denotes averaging with the measure of the field theory path integral. On the path integral we now perform the change of variables corresponding to a spin s' 'symmetry'. Let $J^{(s')}_\mu$ denote the corresponding current. When $J^{(s')}_\mu$ is conserved this change of variables leaves the path integral unchanged in the neighborhood of x (it acts on the insertions, but we ignore those as they are well separated from x). When the current is not conserved, however, it changes the action by $\epsilon\, \partial^\mu J^{(s')}_\mu(y)$. Let us suppose that

$$\partial^\mu J^{(s')}_\mu(y) = \frac{1}{2} \sum_{s_1,s_2} J^{(s_1)} \mathcal{D}^{(s')}_{s_1 s_2} J^{(s_2)} + \cdots\,, \tag{5.71}$$

where $\mathcal{D}^s_{s_1 s_2}$ is a differential operator, It follows that, in the large N limit, the change in the path integral induced by this change of variables is given by

$$\int d^3y \, \langle J^{(s_1)}(y) \cdots \rangle \, \mathcal{D}^{(s')}_{s_1 s} \langle J^s(0) J^{(s)}(y) \rangle$$

(where we have used the fact that the insertion of canonically normalized double trace operator contributes in the large N limit only under conditions of maximal factorization). In other words the symmetry transformation amounts to an effective operator insertion of $J^{(s_1)}$. Specializing to the case $s_1 = 0$ we conclude that, in the presence of a spin s source $J^{(s)}$, a spin s' symmetry transformation should turn on a non normalizable mode for the scalar field given by

$$\mathcal{D}^{(s')}_{0s} \langle J^s(0) J^{(s)}(y) \rangle. \tag{5.72}$$

Before proceeding with our analysis, we pause to restate our derivation of (5.81) in bulk rather than field theory language. Denote collectively by Φ all bulk fields, and by $\varphi^{(s)}_{\mu\cdots}$ a particular bulk field of some spin s. Consider the spin-s' symmetry generated by gauge parameter $\epsilon(x)$, under which $\varphi_{\mu\cdots} \to \varphi_{\mu\cdots} + \delta_\epsilon \varphi_{\mu\cdots}$. Let $\phi(\vec{x})$ be the renormalized boundary value of $\varphi(\vec{x}, z)$, namely $\varphi(\vec{x}, z) \to z^\Delta \phi(\vec{x})$ as $z \to 0$. Let us consider the expectation value of $\phi(\vec{x})$ at the presence of some boundary source $j^{\mu\cdots}$ (of some other spin s) located away from $\vec{x}$. The path integral is invariant under an infinitesimal field redefinition $\Phi \to \Phi + \delta_\epsilon \Phi$, where δ_ϵ takes the form of the asymptotic symmetry variation in the bulk, but vanishes for z less than a small cutoff near the boundary, so as to preserve the prescribed boundary condition, $\Phi(\vec{x}', z) \to z^{3-\Delta} j(\vec{x}') + \mathcal{O}(z^\Delta)$. From this we can write

$$\begin{aligned} 0 &= \int D\Phi \Big|_{\Phi(\vec{x}',z)\to z^{3-\Delta} j(\vec{x}')+\mathcal{O}(z^\Delta)} \delta_\epsilon \left[\varphi^{(s_1)}(\vec{x}, z) \exp\left(-S[\Phi]\right) \right] \\ &= \left\langle \delta_\epsilon \varphi^{(s_1)}(\vec{x}, z) \right\rangle_j - \left\langle \varphi^{(s_1)}(\vec{x}, z) \, \delta_\epsilon S \right\rangle_j . \end{aligned} \tag{5.73}$$

The spin-s source j is subject to the transversality condition $\partial_{i_1} j_{(s)}^{i_1 \cdots i_s} = 0$. Now $\delta_\epsilon S$ should reduce to a boundary term,

$$\delta_\epsilon S = \int_{\partial AdS} dy \, \epsilon \, \partial^\mu J_\mu^{(s')}(y) = \frac{1}{2} \int_{\partial AdS} \epsilon \sum_{s_1, s_2} \phi^{(s_1)} \mathcal{D}^{s'}_{s_1 s_2} \phi^{(s_2)} + \cdots , \tag{5.74}$$

where $\mathcal{D}^s_{s_1 s_2}$ is a differential operator, and J_μ is the boundary current associated with the global symmetry generating parameter ϵ which is now a constant along the cutoff surface, which is then taken to $z \to 0$. On the RHS of (5.74), we omitted possible higher order terms in the fields. From (5.73) we then obtain the relation

$$\begin{aligned} \left\langle \delta_\epsilon \varphi^{(s_1)}(\vec{x}, z) \right\rangle_j &= \left\langle \varphi^{(s_1)}(\vec{x}, z) \int_{\partial AdS} d\vec{x}' \epsilon \, \phi^{(s_1)}(\vec{x}') \mathcal{D}^s_{s_1 s_2} \phi^{(s_2)}(\vec{x}') \right\rangle_j + (\text{higher order}) \\ &= \epsilon \int_{\partial AdS} d\vec{x}' \left\langle \varphi^{(s_1)}(\vec{x}, z) \phi^{(s_1)}(\vec{x}') \right\rangle \mathcal{D}^s_{s_1 s_2} \left\langle \phi^{(s_2)}(\vec{x}') \right\rangle_j + (\text{higher order}). \end{aligned} \tag{5.75}$$

Now specialize to the case $s_1 = 0$, i.e. $\varphi^{(s_1)}$ is the scalar field φ subject to the boundary condition such that the dual operator has dimension Δ. The anomalous symmetry variation shows up in terms of order $z^{3-\Delta}$ in $\delta_\epsilon \varphi(\vec{x}, z)$. After integrating out $\vec{x}'$ using the two-point function of φ and taking the limit $z \to 0$, we obtain the relation

$$\left\langle \delta_\epsilon \varphi(\vec{x}, z) \right\rangle_j \Big|_{z^{3-\Delta}} = \epsilon \, \mathcal{D}^s_{0 s_2} \left\langle \phi^{(s_2)}(\vec{x}) \right\rangle_j + (\text{higher order}), \tag{5.76}$$

Keep in mind that j is the spin-s_2 transverse boundary source, and ϵ is the spin-s global symmetry generating parameter. The differential operator $\mathcal{D}^{s'}_{s_1 s_2}$ appears in the spin-s' current non-conservation relation of the form

$$\partial^\mu J^{(s)}_{\mu \cdots} = \frac{1}{2} \sum_{s_1, s_2} J^{(s_1)}_{\cdots} \mathcal{D}^s_{s_1 s_2} J^{(s_2)}_{\cdots} + (\text{total derivative}) + (\text{triple trace}). \tag{5.77}$$

In particular, the double trace term on the RHS that involves a scalar operator takes the form

$$J^{(0)}(\vec{x}) \, \mathcal{D}^s_{0 s_2} J^{(s_2)}(\vec{x}) + (\text{total derivative}). \tag{5.78}$$

Knowing the LHS of (5.76) from the gauge variation of Vasiliev's bulk fields, and using that fact that $\left\langle \phi^{(s_2)}(\vec{x}) \right\rangle_j$ is given by the boundary two-point function of the spin-s_2 current, we can then derive $\mathcal{D}^s_{0s_2}$ using this Ward identity. In other words we have rederived (5.72).

(5.72) applies to arbitrary sources J^s and also to arbitrary spin s' symmetry transformations. Let us assume that our sources is of the form specified in the previous subsection; all spinor indices on the source are dotted so with a constant spinor λ which is chosen so that

$$\lambda \vec{\sigma} \sigma_z \lambda = \vec{\epsilon}'.$$

In other words our source is uniformly polarized in the ϵ direction. Let us also choose the spin s' variation to be generated by the current $J^\mu_{a_1 \ldots a_{2s'-2}} \Lambda_0^{a_1} \ldots \Lambda_0^{a_{2s'-2}}$ with

$$\Lambda_0 \vec{\sigma} \sigma_z \Lambda_0 = \vec{\epsilon}$$

where $\vec{\epsilon}$ is a constant vector. In other words we have chosen to specialize attention to those symmetries generated by the spin s' current contracted with $s'-1$ translations in the direction ϵ rather than with a generic conformal killing vector. If we compare with the asymptotic symmetry variation the bulk scalar derived earlier we must set Λ_- to zero and $\Lambda_+ = \Lambda_0$. It follows from the previous subsection that

$$\begin{aligned}\delta B^{(0)} = & \frac{4}{(\vec{x}^2)^{2s_2+1}} \frac{1}{(s-s_2-1)!} \left(\frac{2}{\vec{x}^2} \Lambda_0 \vec{x} \cdot \vec{\sigma} \sigma^z \Lambda_0 \right)^{s-s_2-1} \\ & \times \left[\cos\theta_0 (\lambda \vec{x} \cdot \vec{\sigma} \sigma^z \Lambda_0)^{2s_2} z + i \sin\theta_0 \cdot 2s_2 (\lambda \Lambda_0)(\lambda \vec{x} \cdot \vec{\sigma} \sigma^z \Lambda_0)^{2s_2-1} z^2 + \mathcal{O}(z^3) \right].\end{aligned} \tag{5.79}$$

In the $\Delta = 1$ case, the anomalous variation comes from the order z^2 term in (5.79), giving

$$\mathcal{D}^s_{0s_2} \left\langle \phi^{(s_2)}(\vec{x}) \right\rangle_j = \sin\theta_0 C_{ss_2} \frac{(\varepsilon \cdot x)^{s-s_2} (2x \cdot \varepsilon x \cdot \varepsilon' - x^2 \varepsilon \cdot \varepsilon')^{s_2-1} \epsilon^{\mu\nu\rho} \varepsilon'_\mu \varepsilon_\nu x_\rho}{(\vec{x}^2)^{s+s_2+1}}, \tag{5.80}$$

Here C_{ss_2} is a numerical constant that depends only on s and s_2.

(5.80) gives a formula for the appropriate term in (5.71) when the operators that appear in this equation have two point functions

$$\begin{aligned} \langle O(0)O(x)\rangle &= \frac{\alpha_0}{x^2}, \\ \langle J^s(0)J^s(x)\rangle &= \frac{\alpha_s x_-^{2s}}{x^{4s+2}}. \end{aligned} \tag{5.81}$$

Note in particular that these two point functions are independent of the phase θ. Let us now compare this relation to the results of Maldacena and Zhiboedov [51]. Those authors determined the non-conservation relation of currents of spin s, which in the lightcone direction to take the form

$$\partial_\mu J^{(s)\mu}{}_{-\cdots -} = \frac{\widetilde{\lambda}_b}{\sqrt{1+\widetilde{\lambda}_b^2}} \sum_{s'} a_{ss'}\, \epsilon_{-\mu\nu} J^{(0)} \partial_-^{s-s'-1} \partial^\mu J^{(s')\nu}{}_{-\cdots -} + \cdots , \tag{5.82}$$

where $\cdots$ stands for double trace terms involving two currents of nonzero spins, total derivatives, and triple trace terms. Note that the first term we exhibited on the RHS of (5.82) is not a primary by itself, but when combined with the total derivatives term in $\cdots$ becomes a double trace primary operator in the large N limit. We have used the notation $\widetilde{\lambda}_b$ of [51] in the case of quasi-boson theory, but normalized the two-point function of $J^{(0)}$ to be independent of $\widetilde{\lambda}_b$.

Indeed with $(\mathcal{D}^s_{0s'} J^{(s')})_{-\cdots -} \sim \epsilon_{-\mu\nu} \partial_-^{s-s'-1} \partial^\mu J^{(s')\nu}{}_{-\cdots -}$, and the identification

$$\widetilde{\lambda}_b = \tan\theta_0, \tag{5.83}$$

the structure of the divergence of the current agrees with (5.80) obtained from the gauge transformation of bulk fields.

Similarly, in the $\Delta = 2$ case, the anomalous variation comes from the order z term in (5.79). We have

$$\mathcal{D}^s_{0s_2} \left\langle \phi^{(s_2)}(\vec{x}) \right\rangle_j = \cos\theta_0 \widetilde{C}_{ss_2} \frac{(\varepsilon\cdot x)^{s-s'}(2x\cdot\varepsilon x\cdot\varepsilon' - x^2\varepsilon\cdot\varepsilon')^{s'}}{(\vec{x}^2)^{s+s'+1}}. \tag{5.84}$$

This should be compared to the current non-conservation relation in the quasi-fermion theory, of the form

$$\partial_\mu J^{(s)\mu}{}_{-\dots-} = \frac{\widetilde{\lambda}_f}{\sqrt{1+\widetilde{\lambda}_f^2}} \sum_{s'} \widetilde{a}_{ss'} J^{(0)} \partial_-^{s-s'-1} J^{(s')}{}_{-\dots-} + (\text{total derivative}) + \cdots\,, \tag{5.85}$$

Once again, this agrees with the structure of (5.84), with $(\mathcal{D}^s_{0s'} J^{(s')})_{-\dots-} \sim \partial_-^{s-s'-1} J^{(s')}{}_{-\dots-}$, and the identification

$$\widetilde{\lambda}_f = \cot\theta_0. \tag{5.86}$$

Following the argument of [51], the double trace terms involving a scalar operator in the current non-conservation relation we derived from gauge transformation in Vasiliev theory allows us to determine the violation of current conservation in the three-point function, $\left\langle (\partial \cdot J^{(s)})\, J^{(s')} J^{(0)} \right\rangle$, and hence fix the normalization of the parity odd term in the $s - s' - 0$ three-point function.

Here we encounter a puzzle, however. By the Ward identity argument, we should also see an anomalous variation under global higher spin symmetry of a field $\varphi^{(s_1)}$ of spin $s_1 > 1$. This is not the case for our $\delta_\epsilon B^{(s_1)}$ as computed in (5.67). Presumably the resolution to this puzzle lies in the gauge ambiguity in extracting the correlators from the boundary expectation value of Vasiliev's master fields, which has not been properly understood thus far. This gauge ambiguity may also explain why one seems to find vanishing parity odd contribution to the three point function by naively applying the gauge function method of [34].[20]

[20] We thank S. Giombi for discussions on this.

Anomalous higher spin symmetry variation of spin-1 gauge fields

Since one can choose a family of mixed electric-magnetic boundary conditions on the spin-1 gauge field in AdS_4, such a boundary condition will generically be violated by the nonlinear asymptotic higher spin symmetry transformation as well.

Let us consider the self-dual part of the spin-1 field strength, whose variation is given in terms of $\delta_\epsilon B^{(2,0)}(\vec{x},z|y)$, i.e. the terms in $\delta_\epsilon B$ of order y^2 and independent of $\bar{y}$. According to (5.68), the leading order terms in z, namely order z^2 terms, of $\delta_\epsilon B^{(2,0)}(\vec{x},z)$ in the presence of a spin-s boundary source at $\vec{x}=0$ is given by

$$\begin{aligned}\delta_\epsilon B^{(2,0)}(\vec{x},z|y) \;\longrightarrow\; & -\frac{z^2}{|x|^{4s+2}}\left[2\left(\frac{1}{|x|^2}\Lambda_+\sigma^z\mathbf{x}+\Lambda_-\right)y\right]^2 \sinh\left[\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+ - 2\Lambda_+\Lambda_-\right] \\ & \times\left[e^{i\theta_0}(\lambda\mathbf{x}\sigma^z\Lambda_+)^{2s} + e^{-i\theta_0}(\lambda\sigma^z\mathbf{x}\Lambda_+)^{2s}\right] \\ & - e^{i\theta_0}\frac{4sz^2}{|x|^{4s+2}}\cdot\left[2\left(\frac{1}{|x|^2}\Lambda_+\sigma^z\mathbf{x}+\Lambda_-\right)y\right]\cosh\left[\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+ - 2\Lambda_+\Lambda_-\right](\lambda\mathbf{x}\sigma^z y)(\lambda\mathbf{x}\sigma^z\Lambda_+)^{2s-1} \\ & - e^{i\theta_0}\frac{2s(2s-1)z^2}{|x|^{4s+2}}\sinh\left[\frac{2}{x^2}\Lambda_+\sigma^z\mathbf{x}\Lambda_+ - 2\Lambda_+\Lambda_-\right](\lambda\mathbf{x}\sigma^z y)^2(\lambda\mathbf{x}\sigma^z\Lambda_+)^{2s-2}.\end{aligned} \tag{5.87}$$

The anti-self-dual components, $\delta_\epsilon B^{(0,2)}(\vec{x},z|\bar{y})$, is related by complex conjugation. Note that by the linearized Vasiliev equations with parity violating phase θ_0, $B^{(2,0)}$ and $B^{(0,2)}$ are related to the ordinary field strength $F_{\mu\nu}$ of the vector gauge field by

$$\begin{aligned}B^{(2,0)}(x|y) &= e^{i\theta_0}z^2F^+_{\mu\nu}(x)(\sigma^{\mu\nu})_{\alpha\beta}y^\alpha y^\beta, \\ B^{(0,2)}(x|\bar{y}) &= e^{-i\theta_0}z^2F^-_{\mu\nu}(x)(\sigma^{\mu\nu})_{\dot\alpha\dot\beta}\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}.\end{aligned} \tag{5.88}$$

The factor z^2 here comes from the z-dependence of the vielbein in $e^\mu_{\alpha\dot\gamma}e^\nu_{\beta\dot\delta}\epsilon^{\dot\gamma\dot\delta}$. The two point functions of the operators dual to the gauge field in the equation above are given by

$$\langle J^\mu(0)J^\nu(x)\rangle = \frac{1}{\pi^2g^2}\frac{\delta^{\mu\nu}-\frac{2x^\mu x^\nu}{x^2}}{x^4}, \tag{5.89}$$

where g is the bulk gauge coupling constant. The mixed boundary condition

$$F_{ij} = i\zeta\epsilon_{ijk}F_{zi} \quad \text{at } z = 0$$

is equivalent to[21]

$$e^{-i\rho}F^{+}_{zi}\big|_{z=0} = e^{i\rho}F^{-}_{zi}\big|_{z=0}\,, \quad \text{where } \; e^{2i\rho} \equiv \frac{1+i\zeta}{1-i\zeta}. \tag{5.90}$$

We see that precisely when $\theta_0 = 0$ or $\pi/2$, the standard magnetic boundary condition, i.e. $\rho = 0$ ($k = \infty$), is consistent with higher spin gauge symmetry. For generic θ_0, however, there is *no* choice of ρ for the boundary condition to be consistent with the higher spin symmetry variation on $\delta_\epsilon B^{(2,0)}$ and $\delta_\epsilon B^{(0,2)}$. Therefore, we see again that the parity violating phase breaks all higher spin symmetries. From this one can also derive the double trace term involving a spin-1 current in the divergence of the spin-s current of the boundary theory, using the method of the previous subsection.

5.4 Partial breaking of supersymmetry by boundary conditions

In this very important section we now turn to supersymmetric Vasiliev theory. We investigate the action of asymptotic supersymmetry transformations on bulk fields of spin 0, 1/2, and 1. As in the case of higher spin symmetries, we find that no supersymmetry transformation preserves generic boundary conditions. In other words generic boundary conditions on fields violate all supersymmetries. However we identify special classes of

[21] In order to see this let us, for instance, take the special case $i = 1$. The relation becomes $e^{i\rho}(F_{z1} - F_{23}) = e^{-i\rho}(F_{z1} + F_{23})$, so that $F_{23} = \frac{e^{2i\rho}-1}{e^{2i\rho}+1}F_{z1}$.

boundary conditions that that preserve $\mathcal{N} = 1, 2, 3, 4$ and 6 supersymmetries[22] in the next section. We go on present conjectures for CFT duals for these theories.

We emphasize that the boundary conditions presented in this section preserve supersymmetry when acting on *linearized* solutions of Vasiliev's theory. The study of arbitrary linearize solutions is insufficient to completely determine the boundary conditions that preserve supersymmetry as we now explain.

Consider a linearized solution of a bulk scalar dual to an operator of dimension unity. The solution to such a scalar field decays at small z like $\mathcal{O}(z)$, and the boundary condition on this scalar asserts the vanishing of the $\mathcal{O}(z^2)$ term. However terms quadratic in $\mathcal{O}(z)$ are of $\mathcal{O}(z^2)$ at leading order, and so could potentially violate the boundary condition. It follows that the linearized boundary conditions studied presented in this section are not exact, but will be corrected at nonlinear order. Indeed we know one source of such corrections; the boundary condition deformations dual to the triple trace deformations of the dual boundary Chern Simons theory. We ignore all such nonlinear deformations in this section (see the next section for some remarks).

5.4.1 Structure of Boundary Conditions

Consider the n-extended supersymmetric Vasiliev theory with parity violating phase θ_0. We already know that all higher spin symmetries are broken by any choice of boundary condition on fields of low spins, as expected for any interacting CFT. We also expect that any parity non-invariant CFT to have at most $\mathcal{N} = 6$ supersymmetry, and the question is

[22] Theories with $\mathcal{N} = 5$ supersymmetry involve SO and Sp gauge groups on the boundary. Such theories presumably have bulk duals in terms of the 'minimal' Vasiliev theory, which we, however, never study in this paper. We thank O. Aharony and S. Yokoyama for related discussions.

whether the breaking of supersymmetries to $\mathcal{N} \leq 6$ in the n-extended Vasiliev theory can be seen from the violating of boundary conditions by supersymmetry variations. The answer will turn out to be yes. In fact, we will be able to identify boundary conditions that preserve $\mathcal{N} = 0, 1, 2, 3, 4$ and 6 supersymmetries, in precise agreement with the various $\mathcal{N}$-extended supersymmetric Chern-Simons vector models that differ from one another by double and triple trace deformations.

To begin we shall describe a set of boundary condition assignments on all bulk fields of spin 0, $\frac{1}{2}$, and 1, that will turn out to preserve various number of supersymmetries and global flavor symmetries. The supersymmetry transformation of the bulk fields of spin 0, $\frac{1}{2}$, and 1 are derived explicitly in terms of the master field $B(x|Y)$ in Appendix 5.B. For convenience we will speak of the n-extended parity violating supersymmetric Vasiliev theory with no extra Chan-Paton factors, though our discussion can be straightforwardly generalized to include $U(M)$ Chan-Paton factors. The bulk theory together with the prescribed boundary conditions are then conjectured to be holographically dual to supersymmetric Chern-Simons vector models with various number of supersymmetries and superpotentials.

Scalars

Vasiliev's theory contains 2^{n-2} parity even scalar fields and an equal number of parity odd scalar fields. We expect the most general allowed boundary condition for these fields to take the form (5.121) (with d_{abc} set to zero, as we restrict attention to linear analysis in this section). If we view the collection of scalar fields as a linear vector space of dimension 2^{n-1} then (5.121) asserts that the z component of scalars lies in a particular half dimensional subspace of this vector space, while the z^2 component of the scalars lies in a complementary half dimensional subspace (obtained from the first space by switching the role of parity even and parity odd scalars). Now the Vasiliev master field B packs all 2^{n-1} scalars into a single even function of ψ_i. In order to specify the boundary conditions on scalars, we must specify the 2^{n-2} dimensional subspace (of the 2^{n-1} dimensional space of even functions of ψ^i) that multiply z in the small z expansion of these fields. We must also choose out a half dimensional subspace of functions that multiply z^2 (as motivated above, this subspace will always turn out to be complementary to the first).

How do we specify the subspaces of interest? The technique we adopt is the following. We choose any convenient reference subspace S that has the property that $S + \Gamma S$ is the full space. Let γ be an arbitrary hermitian operator (built out of the ψ_i fields) that acts on the subspace S - i.e. Γ is the exponential of a linear combination of projectors for the basis states of S. An arbitrary real half dimensional subspace in the space of functions is given by $e^{i\gamma}S + \Gamma e^{-i\gamma}S$. The complementary subspace (obtained by flipping parity even and parity odd functions) is given by $e^{i\gamma}S - \Gamma e^{-i\gamma}S$. In other words the most general boundary conditions for the scalar part of B takes the form

$$B^{(0)}(\vec{x}, z) = (e^{i\gamma} + \Gamma e^{-i\gamma})\tilde{f}_1(\psi)z + (e^{i\gamma} - \Gamma e^{-i\gamma})\tilde{f}_2(\psi)z^2 + \mathcal{O}(z^3) \tag{5.91}$$

where $f_1(\psi)$ and $f_2(\psi)$ represent any function - not necessarily the same - that lie within the reference real half dimensional subspace on the space of functions of ψ, and γ is an operator, to be specified, that acts on this subspace. It is not difficult to verify that (5.91) is consistent with the reality of B. (5.91) may also be rewritten as

$$\begin{aligned} B^{(0)}(\vec{x}, z) = z &\Big((1+\Gamma)\cos\gamma\,\tilde{f}_1 + (1-\Gamma)i\sin\gamma\,\tilde{f}_1\Big) \\ &+ z^2\Big((1-\Gamma)\cos\gamma\,\tilde{f}_2 + (1+\Gamma)i\sin\gamma\,\tilde{f}_2\Big) + \mathcal{O}(z^3), \end{aligned} \tag{5.92}$$

a form that makes the connection with (5.121) more explicit.

In the special case $\gamma = 0$, $\tilde{f}_1$ and $\tilde{f}_2$ can be arbitrary (i.e. the reference half dimensional space can be chosen arbitrarily) and (5.91) simply asserts that parity odd scalars have dimension 1 while parity even scalars have dimension 2.

Spin half fermions

Boundary conditions for spin half fermions are specified more simply than for their scalar counterparts. The most general boundary condition relates the parity even part of any given fermion (the 'source') to the parity odd piece of all other fermions ('the vev'). The most general real boundary condition of this form is that the spin-$\frac{1}{2}$ part of B take the form

$$B^{(\frac{1}{2})}(\vec{x}, z|Y)\big|_{\mathcal{O}(y,\bar{y})} = z^{\frac{3}{2}}\left[e^{i\alpha}(\chi y) - \Gamma e^{-i\alpha}(\bar{\chi}\bar{y})\right] + \mathcal{O}(z^{\frac{5}{2}}), \qquad \chi = \sigma^z\bar{\chi}. \tag{5.93}$$

where χ is an arbitrary spinor and α is an arbitrary hermitian operator (i.e. function of ψ_i). Reality of $B^{(\frac{1}{2})}$ imposes $(\chi^\alpha)^* = -i\bar{\chi}_{\dot{\alpha}}$.

In the limit $\alpha = 0$ these boundary conditions simply assert that the $z^{\frac{3}{2}}$ fall off of the fermion is entirely parity odd. Recall that according to the standard AdS/CFT rules, the parity even component of the fermion field may be identified with the expectation value of the boundary operator, while the parity odd part is an operator deformation. When α (which in general is a linear operator that acts on $\chi, \bar{\chi}$, which are functions of ψ) is nonzero, the boundary conditions assert a linear relation between parity even and parity odd pieces, of the sort dual to a fermion-fermion double trace operator.

Gauge Fields

The electric-magnetic mixed boundary condition on the spin-1 field is

$$B^{(1)}(\vec{x}, z|Y)\big|_{\mathcal{O}(y^2,\bar{y}^2)} = z^2\left[e^{i\beta}(yFy) + \Gamma e^{-i\beta}(\bar{y}\overline{F}\bar{y})\right] + \mathcal{O}(z^3), \qquad F = -\sigma^z\overline{F}\sigma^z. \tag{5.94}$$

Here β is equal to θ_0 for the magnetic boundary condition, corresponding to ungauged flavor group in the boundary CFT (recall that $e^{i\theta}F$ is identified with the bulk Maxwell field strength; see above). Once again β is, in general, an operator that acts on $F, \overline{F}$. Reality of $B^{(1)}$ gives $(F^{\alpha}_{\beta})^* = \bar{F}^{\dot{\beta}}_{\dot{\alpha}}$

We will see that the $\mathcal{N} = 4$ and $\mathcal{N} = 6$ boundary conditions requires taking β to be a nontrivial linear operator that acts on $F, \overline{F}$, which amounts to gauging a flavor group with a finite Chern-Simons level.

Now to characterize the boundary condition, we simply need to give the linear operators α, γ, β which act on $\tilde{f}_{1,2}(\psi)$, $\chi(\psi)$, $F(\psi)$, and a set of linear conditions on $\tilde{f}_{1,2}(\psi)$.

We now proceed to enumerate boundary conditions that preserve different degrees of supersymmetry. In each case we also conjecture a field theory dual for the resultant Vasiliev theory. For future use we present the Lagrangians of the corresponding field theories in Appendix 5.D.

5.4.2 The $\mathcal{N} = 2$ theory with two $\square$ chiral multiplets

Let us start with $n = 4$ extended supersymmetric Vasiliev theory. The master fields depend on the auxiliary Grassmannian variables $\psi_1, \psi_2, \psi_3, \psi_4$. With $\theta(X) = 0$, $\alpha = 0$ and $\gamma = 0$ in the fermion and scalar boundary conditions, respectively, the dual CFT is the free theory of 2 chiral multiplets (in $\mathcal{N} = 2$ language) in the fundamental representation of $SU(N)$, with a total number of 16 supersymmetries. Now we will turn on nonzero θ_0, and describe a set of boundary conditions that preserve $\mathcal{N} = 2$ supersymmetry (4 supercharges) and $SU(2)$ flavor symmetry. The boundary condition for the spin-1 field is the standard magnetic one. The boundary condition for spin-$\frac{1}{2}$ and spin-0 fields are given by (5.235), (5.236), (5.243), with

$$\alpha = \gamma = \theta_0, \quad [\psi_1, \tilde{f}_1] = [\psi_1, \tilde{f}_2] = 0 \ \text{ or } \ P_{1,\psi_2\psi_3,\psi_2\psi_4,\psi_3\psi_4}\tilde{f}_{1,2} = \tilde{f}_{1,2}. \tag{5.95}$$

where $P_{\psi_i,\cdots}$ stands for the projection onto the subspace spanned by the monomials $\psi_i, \cdots$; $\tilde{f}_{1,2}$ are subject to the constraint that they commute with ψ_1, or equivalently, $\tilde{f}_{1,2}$ are spanned by $1, \psi_2\psi_3, \psi_2\psi_4, \psi_3\psi_4$. The 2 supersymmetry parameters are given by $\Lambda_+ = \Lambda_0$, $\Lambda_- = 0$, with

$$\Lambda_0 = \eta\psi_1 \ \text{ and } \ \eta\psi_1\Gamma, \tag{5.96}$$

where $\Gamma = \psi_1\psi_2\psi_3\psi_4$. η is a constant Grassmannian spinor parameter that anti-commutes with all ψ_i's.

Clearly, with $\alpha = \theta_0$, (5.234) obeys the fermion boundary condition (5.235), (5.236), and (5.241) obeys the magnetic boundary condition on the spin-1 fields (5.226), (5.227). (5.242) with $\alpha = \gamma$ obeys (5.243) with $\tilde{f}_{1,2}$ of the form $\{\psi_1, \lambda\}$, or $\{\psi_1\Gamma, \lambda\}$, both of which commute with ψ_1. Finally, in the RHS of (5.246), all commutators of $\tilde{f}_{1,2}$ vanish, leaving the terms with anti-commutators only, which satisfy (5.267), (5.236) with $\gamma = \alpha$. Clearly, an $SU(2) \simeq SO(3)$ flavor symmetry rotating ψ_2, ψ_3, ψ_4 is preserved by this $\mathcal{N} = 2$ boundary condition.

It is natural to propose that the $n = 4$ extended parity violating Vasiliev theory with this boundary condition is dual to $\mathcal{N} = 2$ Chern-Simons vector model with 2 fundamental chiral multiplets. There is no gauge invariant superpotential in this case, while there is an $SU(2)$ flavor symmetry[23] rotating the two chiral multiplets, which is identified with the

[23] Note that the field theory is left invariant under a larger set of $U(2)$ transformations, which rotates the chiral multiplets into each other. However the diagonal $U(1)$ in $U(2)$ acts in the same way on all fundamental fields, and so is part of the $U(N)$ gauge symmetry. There is nonetheless a bulk gauge field - with ψ content I -formally corresponding to this $U(1)$ factor.

$SO(3)$ symmetry of rotations in ψ_1, ψ_2 and ψ_3 preserved by the boundary conditions listed above.

Let us elaborate on, for instance, the scalar boundary conditions. There are a total of eight scalars in the problem (the number of even functions of ψ_i). A basis for parity even scalars is given by $(1+\Gamma)$ and $(1+\Gamma)\psi_1\psi_i$ where $i = 1 \ldots 3$. A basis for parity odd scalars is given by $(1-\Gamma)$ and $(1-\Gamma)\psi_1\psi_i$. In each case the scalars transform in the $1+3$ of $SU(2)$. Recall that the fundamental fields of the field theory (scalars as well as fermions) transform in the $\frac{1}{2}$ of the flavour symmetry $SU(2)$; it follows that bilinears in these fields also transform in the $1+3$ of $SU(2)$, establishing a natural map between bulk fields and field theory operators.

The boundary conditions (5.95) assert that the coefficient of the $\mathcal{O}(z^2)$ term of the parity even scalars/vectors is equal to $\tan\theta_0$ times the coefficient of the $\mathcal{O}(z^2)$ of the corresponding parity odd scalars/vectors. Similarly the coefficient of the $\mathcal{O}(z)$ term of the parity odd scalars/vectors is equal to $\tan\theta_0$ times the coefficient of the $\mathcal{O}(z)$ of the corresponding parity even scalars/vectors. This is exactly the kind of boundary condition generated by a double trace deformation that couples the dual dimension one and dimension two operators, with equal couplings in the scalar and vector (of $SU(2)$) channels. We will elaborate on this in much more detail in the next section.

5.4.3 A family of $\mathcal{N}=1$ theories with two $\square$ chiral multiplets

If we keep only the supersymmetry generator given by

$$\Lambda_0 = \eta\psi_1, \tag{5.97}$$

then a one-parameter family of boundary conditions that preserve $\mathcal{N}=1$ supersymmetry is given by

$$\alpha = \theta_0 P_1^S + \gamma P_1^A, \qquad \beta = \theta_0, \quad [\psi_1, \tilde{f}_1] = [\psi_1, \tilde{f}_2] = 0, \tag{5.98}$$

where P_1^S and P_1^A are the projection operators that projects an odd function of ψ_i's onto the subspaces spanned by

$$\psi_1\Gamma, \psi_2, \psi_3, \psi_4 \qquad (\text{all anti}-\text{commute with } \psi_1) \tag{5.99}$$

and

$$\psi_1, \psi_2\Gamma, \psi_3\Gamma, \psi_4\Gamma \qquad (\text{all commute with } \psi_1) \tag{5.100}$$

respectively. γ is now an arbitrary phase (independent of ψ_i).

This family of boundary conditions is dual to $\mathcal{N}=1$ deformations of the $\mathcal{N}=2$ theory with two chiral flavors, by turning on an $\mathcal{N}=1$ (non-holomorphic) superpotential that preserves the $SU(2)$ flavor symmetry (corresponding to the bulk symmetry that rotates ψ_2, ψ_3, ψ_4).

The same theory can also be rewritten as the $n=2$ extended supersymmetric Vasiliev theory with $M=2$ matrix extension. The spin-1, fermion, and scalar boundary conditions are given by

$$\alpha = \theta_0 P_{\psi_2} + \gamma P_{\psi_1}, \qquad \beta = \theta_0, \qquad [\psi_1, \tilde{f}_1] = [\psi_1, \tilde{f}_2] = 0. \tag{5.101}$$

It is natural to wonder about the relationship between the parameter γ above and the field theory parameter ω (see (5.300)). General considerations leave this relationship undetermined; however we will present a conjecture for this relationship in the next section.

5.4.4 The $\mathcal{N}=2$ theory with a $\square$ chiral multiplet and a $\overline{\square}$ chiral multiplet

Now let us describe a boundary condition that preserve the two supersymmetries generated by

$$\Lambda_-=0, \quad \Lambda_0=\eta\psi_1 \quad \text{and} \quad \eta\psi_2. \tag{5.102}$$

It is given by

$$\beta=\theta_0, \quad \alpha=\theta_0(1-P_{\psi_3\Gamma,\psi_4\Gamma}), \quad \gamma=\theta_0 P_{1,\psi_3\psi_4}. \tag{5.103}$$

where $P_{\psi_i,\cdots}$ stands for the projection onto the subspace spanned by the monomials $\psi_i,\cdots$, as before; $\tilde{f}_{1,2}$ are now subject to the constraint that they commute with *either* ψ_1 *or* ψ_2, i.e. $\tilde{f}_{1,2}$ are spanned by $1,\psi_3\psi_4,\ \psi_1\psi_3,\ \psi_1\psi_4,\ \psi_2\psi_3,\ \psi_2\psi_4$. Note that when acting on the latter four monomials, γ vanishes, and $\tilde{f}_1$ and $\tilde{f}_2$ may be replaced by $\frac{1+\Gamma}{2}\tilde{f}_1$ and $\frac{1-\Gamma}{2}\tilde{f}_2$. Therefore, only half of the components of $\tilde{f}_{1,2}$ are independent, as required. One can straightforwardly verified that this set of boundary conditions preserve the two supersymmetries (5.102). Clearly, the $U(1)$ flavor symmetry that rotates ψ_3,ψ_4 is still preserved, but there is no $SU(2)$ flavor symmetry. We also have the $U(1)$ R symmetry corresponding to rotations of ψ_1,ψ_2.

The $n=4$ Vasiliev theory with this boundary is then naturally proposed to be dual to $\mathcal{N}=2$ Chern-Simons vector model with a fundamental and an anti-fundamental chiral flavor, with $U(1)\times U(1)$ flavor symmetry [24] (corresponding to the components of the bulk vector gauge field proportional to 1 and $\psi_3\psi_4$) besides the $U(1)$ R-symmetry, which means that the $\mathcal{N}=2$ superpotential vanishes, since a nonzero superpotential would break the

[24] One of these two $U(1)$ factors is actually part of the gauge group and so acts trivially on all gauge invariant operators.

$U(1) \times U(1)$ flavor symmetry to a single $U(1)$.

5.4.5 A family of $\mathcal{N} = 2$ theories with a $\square$ chiral multiplet and a $\overline{\square}$ chiral multiplet

The boundary condition in the above section is a special point inside a one-parameter family of boundary conditions which preserved the same set of supersymmetries. It is given by

$$\begin{aligned} &\beta = \theta_0, \quad \alpha = \theta_0(1 - P_{\psi_3\Gamma,\psi_4\Gamma}) + \tilde{\alpha}(P_{\psi_3\Gamma} - P_{\psi_4\Gamma}), \\ &\gamma = \theta_0 P_{1,\psi_3\psi_4} + \tilde{\alpha} P_{\psi_2\psi_4,\psi_1\psi_4}, \\ &P_{1,\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4} \tilde{f}_{1,2} = \tilde{f}_{1,2}. \end{aligned} \tag{5.104}$$

This one-parameter family of deformations is naturally identified with the superpotential deformation of the $\mathcal{N} = 2$ Chern-Simons vector model with a fundamental and an anti-fundamental chiral flavor. This superpotential is marginal at infinite N; at finite N there are two inequivalent conformally invariant fixed points [72]. The $\tilde{\alpha} = 0$ point is the boundary condition on the above section, describing the $\mathcal{N} = 2$ theory with no superpotential, whereas $\tilde{\alpha} = \pm\theta_0$ give the $\mathcal{N} = 3$ point, as will be discussed in the next subsection.

5.4.6 The $\mathcal{N} = 3$ theory

The $\mathcal{N} = 3$ boundary condition that preserve supersymmetry generated by the parameters

$$\Lambda_- = 0, \quad \Lambda_0 = \eta\psi_1, \ \eta\psi_2, \ \text{and } \eta\psi_3, \tag{5.105}$$

is given by

$$\beta = \theta_0, \quad \alpha = \theta_0(1 - P_{\psi_1\psi_2\psi_3}) - \theta_0 P_{\psi_1\psi_2\psi_3}, \quad \gamma = \theta_0, \quad P_{1,\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}\tilde{f}_{1,2} = \tilde{f}_{1,2}. \tag{5.106}$$

This boundary condition is dual to the $\mathcal{N} = 3$ Chern-Simons vector model with a single fundamental hypermultiplet, which may be obtained from the $\mathcal{N} = 2$ theory with a fundamental and an anti-fundamental chiral multiplet by a turning on a superpotential. The $SO(3)$ symmetry of rotations in ψ_1, ψ_2 and ψ_3 maps to the $SO(3)$ R-symmetry of the model. Notice that unlike the case studied in Section 5.4.2, $\alpha \neq \gamma$ reflecting the fact that the $SO(3)$ R symmetry, unlike a flavor symmetry, acts differently on bosons and fermions.

5.4.7 The $\mathcal{N} = 4$ theory

The $\mathcal{N} = 4$ boundary condition that preserve supersymmetry generated by the parameters

$$\Lambda_- = 0, \quad \Lambda_0 = \eta\psi_i, \quad i = 1, 2, 3, 4, \tag{5.107}$$

is given by

$$\beta = \theta_0(1 - P_\Gamma), \quad \alpha = \theta_0 P_{\psi_i}, \quad \gamma = \theta_0 P_1. \tag{5.108}$$

$\tilde{f}_{1,2}$ are subject to the constraint

$$P_\Gamma \tilde{f}_{1,2} = 0. \tag{5.109}$$

Note also that the components of $\tilde{f}_{1,2}$ proportional to $\psi_i\psi_j$ are subject to the projection $\frac{1\pm\Gamma}{2}$ also, as follows automatically from (5.91), (5.92). The boundary conditions above are invariant under the $SO(4)$ R symmetry of rotations in ψ_1, ψ_2, ψ_3 and ψ_4.

This boundary condition is dual to the $\mathcal{N} = 4$ Chern-Simons quiver theory with gauge group $U(N)_k \times U(1)_{-k}$ and a single bi-fundamental hypermultiplet. The latter can be obtained from the $\mathcal{N}=3$ $U(N)_k$ Chern-Simons vector model with one hypermultiplet flavor by gauging the $U(1)$ flavor current multiplet with another $\mathcal{N}=3$ Chern-Simons gauge field at level $-k$ [73].

5.4.8 An one parameter family of $\mathcal{N}=3$ theories

There is an one parameter family of boundary conditions that preserves the same supersymmetry as in Section 5.4.6,

$$\begin{aligned}
&\beta=\theta_0(1-P_\Gamma)+\tilde{\beta}P_\Gamma, \quad \alpha=\theta_0 P_{\psi_i}+\tilde{\beta}(P_{\psi_1\Gamma,\psi_2\Gamma,\psi_3\Gamma}-P_{\psi_4\Gamma}),\\
&\gamma=\theta_0 P_1+\tilde{\beta}P_{\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4},\\
&P_{1,\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}\tilde{f}_{1,2}=\tilde{f}_{1,2}.
\end{aligned} \tag{5.110}$$

The boundary condition in Section 5.4.6 is at $\tilde{\beta}=\theta_0$. At $\tilde{\beta}=0$, the (5.110) coincides with (5.108), and the $\mathcal{N}=3$ supersymmetry is enhanced to $\mathcal{N}=4$.

5.4.9 The $\mathcal{N}=6$ theory

To construct the bulk dual of the $\mathcal{N}=6$ ABJ vector model [74, 75], we need to double the number of matter fields in the boundary field theory, and correspondingly quadruple the number of bulk fields. This is achieved with the $n=6$ extended supersymmetric Vasiliev theory, which in the parity even case (dual to free CFT) can have up to 64 supersymmetries. We are interested in the parity violating theory, with nonzero interaction phase θ_0, with a set of boundary conditions that preserve $\mathcal{N}=6$ supersymmetries[25], generated by the parameters

$$\Lambda_0 = \eta\psi_i, \quad i = 1, 2, \cdots, 6. \tag{5.111}$$

Similarly to the $\mathcal{N}=4$ theory with one hypermultiplet, here we need to take the boundary condition on the bulk spin-1 field to be

$$\beta = \theta_0(1 - P_\Gamma) - \theta_0 P_\Gamma. \tag{5.112}$$

The spin-$\frac{1}{2}$ and spin-0 boundary conditions are given by

$$\alpha = \theta_0(1 - P_{\psi_i\Gamma}) - \theta_0 P_{\psi_i\Gamma}, \qquad \gamma = \theta_0 P_{1,\psi_i\psi_j}, \tag{5.113}$$

where $P_{\psi_i\Gamma}$ for instance stands for the projection onto the subspace spanned by *all* $\psi_i\Gamma$'s, $i = 1, 2, \cdots, 6$. $\tilde{f}_{1,2}$ are subject to the constraint

$$P_{\Gamma,\psi_i\psi_j\Gamma}\tilde{f}_{1,2} = 0, \tag{5.114}$$

which projects out half of the components of $\tilde{f}_{1,2}$. Note that these boundary conditions enjoy invariance under the $SO(6)$ R symmetry rotations of the ψ_i coordinates.

By comparing the difference between β and θ_0 with the Chern-Simons level of what would be the flavor group of the $\mathcal{N}=3$ Chern-Simons vector model with two hypermultiplets, we will be able to identify θ_0 in terms of k below.

[25]One can show that there is no boundary condition for the $n > 6$ extended supersymmetric Vasiliev theory that preserves $\mathcal{N} = n$ supersymmetries. We expect that there is no $\mathcal{N} > 6$ boundary condition for the parity violating Vasiliev theory, though we have not proven this in general.

5.4.10 Another one parameter family of $\mathcal{N} = 3$ theories

There is another one parameter family of boundary conditions that preserves the same supersymmetry as in Section 5.4.6,

$$
\begin{aligned}
&\beta = \theta_0(1 - P_\Gamma) + \tilde{\beta} P_\Gamma, \\
&\alpha = \theta_0(P_{\psi_i,\psi_a} + P_{\psi_i\psi_j\psi_a,\psi_i\psi_a\psi_b,\psi_4\psi_5\psi_6} - P_{\psi_a\Gamma}) + \tilde{\beta}(P_{\psi_i\Gamma} - P_{\psi_1\psi_2\psi_3}), \\
&\gamma = \theta_0 P_{1,\psi_i\psi_a,\psi_a,\psi_b} - \tilde{\beta} P_{\psi_i\psi_j}, \\
&P_{1,\psi_i\psi_j,\psi_i\psi_a,\psi_a\psi_b} \tilde{f}_{1,2} = \tilde{f}_{1,2},
\end{aligned}
\tag{5.115}
$$

where $i, j = 1, 2, 3$ and $a, b = 4, 5, 6$. At $\tilde{\beta} = -\theta_0$, the (5.115) coincides with the boundary condition in 5.4.9, and the $\mathcal{N} = 3$ supersymmetry is enhanced to $\mathcal{N} = 6$.

5.5 Deconstructing the supersymmetric boundary conditions

5.5.1 The goal of this section

As we have explained early in this paper, the Vasiliev dual to free boundary superconformal Chern Simons theories is well known. In the previous section we have also conjectured phase and boundary condition deformations of this Vasiliev theory that describe the bulk duals of several fixed lines of superconformal Chern Simons theories with known Lagrangians. These interacting superconformal Chern Simons theories differ from their free counterparts in three important respects.

- 1. The level k of the $U(N)$ Chern-Simons theory is taken to infinity holding $\frac{N}{k} = \lambda$ fixed. The free theory is recovered on taking $\lambda \to 0$.

- 2. The Lagrangian of the theory includes marginal triple trace interactions of the schematic form $(\phi^2)^3$ and double trace deformations of the form $(\phi^2)(\psi^2)$ and $(\phi\psi)^2$ (the brackets indicate the structure of color index contractions).

- 3. In some examples including the $\mathcal{N} = 6$ ABJ theory we will also gauge a subgroup of the global symmetry group of the theory with the aid of a new Chern-Simons gauge field.

In this section we carefully compare the supersymmetric boundary conditions, determined in the previous section, with the Lagrangian of the conjectured field theory duals of these systems. This analysis allows us to understand the separate contributions of each of the three factors listed above to the boundary conditions of the previous section. It also yields some information about the relationship between the bulk deformation parameters and field theoretic quantities.

The analysis presented in this section was partly motivated by the following quantitative goal. In the previous section we have presented two one parameter sets of $\mathcal{N} = 3$ Vasiliev boundary conditions (5.110) and (5.115) at any given fixed value of the Vasiliev phase θ_0. The first of these fixed lines interpolates to an $\mathcal{N} = 4$ theory while the second which interpolates to a $\mathcal{N} = 6$ theory. For each line of boundary conditions we have also conjectured a one parameter set of dual boundary field theories. In order to complete the statement of the duality between these systems we need to propose an identification of the parameter that labels boundary conditions with the parameter that labels the dual field theories. The analysis of this section was undertaken partly in order to establish this map. We have been only partly successful in this respect. While we propose a tentative identification of parameters below, there is an unresolved puzzle in the analysis that leads to this identification;

as a consequence we are not confident of this identification. We leave the resolution of this puzzle to future work.

We begin this lengthy section with a review of well known effects of items (2) and (3) listed above on the bulk dual systems. With these preliminaries out of the way we then turn to the main topic of this section, namely the deconstruction of the supersymmetric boundary conditions determined in the previous subsection.

5.5.2 Marginal multitrace deformations from gravity

As we have reviewed in the previous section, the supersymmetric Vasiliev theory contains fields of every half integer spin, including scalars with $m^2 = -2$, spin half fields with $m = 0$, and massless vectors. It is well known that the only consistent boundary conditions for the fields with spin $s > 1$ is that they decay near $z = 0$ like z^{s+1}.[26] On the other hand consistency permits more interesting boundary conditions for fields of spin zero, spin half and spin one. In this section we will review the subset of these boundary conditions that preserve conformal invariance, together with their dual boundary interpretations. The discussion in this subsection is an application of well known material (see for example the references [64, 76, 77, 78, 79, 65] - we most closely follow the approach of the paper [77]).

scalars

The Vasiliev theories we study contain a set of scalar fields propagating in AdS$_4$, all of which have have $m^2 = -2$ in AdS units. In the free theory the boundary conditions for some of these scalars, S_a, are chosen so that the corresponding operator has dimension 1

[26]In other words the coefficient of the leading fall off is required to vanish.

(these are the so called alternate boundary conditions) while the boundary conditions for the remaining scalars, F_α, are chosen so that its dual operator has dimension 2 (these are the so called regular boundary conditions). See Appendix 5.C.1 for a detailed discussion of these boundary conditions and their dual bulk interpretation.

Let us suppose that the Lagrangian for these scalars at quadratic order takes the form[27]

$$\sum_a \frac{1}{g_a^2} \int \sqrt{g} \left(\partial_\mu \bar{S}_a \partial^\mu S_a - 2\bar{S}_a S_a \right) + \sum_\alpha \frac{1}{g_\alpha^2} \int \sqrt{g} \left(\partial_\mu \bar{F}_\alpha \partial^\mu F_\alpha - 2\bar{F}_\alpha F_\alpha \right). \tag{5.116}$$

The redefinition

$$S_a = g_a s_a, \quad F_\alpha = g_\alpha f_\alpha$$

sets all couplings to unity as in the discussion in Appendix 5.C.1.

As explained in detail in Appendix 5.C.1 the action and boundary conditions of bulk scalars do not completely characterize the boundary dynamics of the system. For instance in a theory with a single regular quantized scalar and one alternately quantized scalar there exist a one parameter set of inequivalent boundary actions, each of which lead to identical boundary conditions for (appropriately redefined) bulk fields. However there is a distinguished 'simplest' set of boundary counterterms corresponding to any particular boundary conditon (this is the undeformed or $\theta_0 = 0$ system described in Appendix 5.C.1). This simple counterterm has the following distinguishing property; it yields vanishing two point functions between any operator of dimension one and any other operator of dimension two. Every other choice of counterterms yields correlators between these operators that vanish at separated points but are have non-vanishing contact term contributions.

[27]Vasiliev's theory is currently formulated in terms of equations of motion rather than an action. As a consequence, the values of the coupling constants g_a and g_α, for the scalars that naturally appear in Vasiliev's equations, are undetermined by a linear analysis. The study of interactions would permit the determination of the relative values of coupling constants, but we do not perform such a study in this paper.

In this section we assume that the counterterm action corresponding to the scalar boundary conditions above takes the simple ($\theta_0 = 0$) form referred to above. We will then deduce the effect of a double and triple trace deformation on the boundary conditions of bulk fields.

The two point functions of the operators dual to s_a and f_α[28] are given by[29] [80][30]

$$\begin{aligned} &\frac{1}{2\pi^2}\frac{1}{x^2} \quad \text{(operators dual to } s_a\text{)}, \\ &\frac{1}{2\pi^2}\frac{2}{x^4} \quad \text{(operators dual to } f_\alpha\text{)}. \end{aligned} \tag{5.117}$$

Later in this paper we will be interested in determining the Vasiliev dual to large N theories deformed by double and triple trace scalar operators. The field theory deformations we study are marginal in the large N limit and take the form

$$\int d^3x \left(\frac{\pi^2}{2k^2} c_{abc}\sigma^a\sigma^b\sigma^c + \frac{2\pi}{k} d_{a\alpha}\sigma^a\phi^\alpha \right) \tag{5.118}$$

where σ^a is proportional to the operator dual to s_a and ϕ^α is proportional to the operator dual to f_α (the factors in (5.118) have been inserted for future convenience). We will assume that it is known from field theoretic analysis that

$$\begin{aligned} \langle \sigma^a(x)\sigma^b(0)\rangle &= \delta^{ab}\frac{2Nh_+^a}{(4\pi)^2x^2}, \\ \langle \phi^\alpha(x)\phi^\beta(0)\rangle &= \delta^{\alpha\beta}\frac{4Nh_-^\alpha}{(4\pi)^2x^4}, \end{aligned} \tag{5.119}$$

[28] i.e. the two point functions for the operators for which coefficient of the z^2 fall off of the field s_a is a source, and the operator for which the coefficient of the z fall off of the field f_α is the source

[29] The general formula for the nontrivial prefactor is $\frac{\Gamma(\Delta+1)(2\Delta-d)}{\pi^{\frac{d}{2}}\ \Gamma(\Delta-d/2)\Delta}$.

[30] The Fourier transforms

$$G(k) = \int d^3x e^{ik.x} G(x)$$

(appropriately regulated) evaluate to $\frac{1}{|k|}$ for the dimension one operator (alternate quantization), and to $-|k|$ for the dimension two operator (regular quantization). Note that these quantities are the negative inverses of each other, in agreement with the general analysis of Appendix 5.C.1.

(the factors on the RHS have been inserted for later convenience; h_+^a and h_-^α are numbers). It follows from a comparison of (5.119) and (5.117) that the operator dual to s^a is $\frac{2}{\sqrt{Nh_+^a}}\sigma^a$ while the operator dual to f^α is $\frac{2}{\sqrt{Nh_-^\alpha}}\phi^\alpha$

Let us suppose that at small z,[31]

$$s_a = s_a^{(1)} z + s_a^{(2)} z^2 + \mathcal{O}(z^3), \quad f_\alpha = f_\alpha^{(1)} z + f_\alpha^{(2)} z^2 + \mathcal{O}(z^3). \tag{5.120}$$

It follows from the analysis of 5.C.1 that the marginal deformation (5.118) induces the boundary conditions

$$\begin{aligned} s_a^{(2)} &= \frac{\pi N\sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha} f_\alpha^{(2)} + 3\frac{\pi^2 N^{\frac{3}{2}}\sqrt{h_+^a h_+^b h_+^c}}{16k^2} c_{abc} s_b^{(1)} s_c^{(1)}, \\ f_\alpha^{(1)} &= -\frac{\pi N\sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha} s_a^{(1)}. \end{aligned} \tag{5.121}$$

If we denote the boundary expansion of the original bulk fields by

$$S_a = S_a^{(1)} z + S_a^{(2)} z^2 + \mathcal{O}(z^3), \quad F_\alpha = F_\alpha^{(1)} z + F_\alpha^{(2)} z^2 + \mathcal{O}(z^3), \tag{5.122}$$

then

$$\begin{aligned} \frac{S_a^{(2)}}{g_a} &= \frac{\pi N\sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha} \frac{F_\alpha^{(2)}}{g_\alpha} + 3\frac{\pi^2 N^{\frac{3}{2}}\sqrt{h_+^a h_+^b h_+^c}}{16k^2} c_{abc} \frac{S_b^{(1)}}{g_b}\frac{S_c^{(1)}}{g_c}, \\ \frac{F_\alpha^{(1)}}{g_\alpha} &= -\frac{\pi N\sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha} \frac{S_a^{(1)}}{g_a}. \end{aligned} \tag{5.123}$$

In summary the boundary conditions (5.123) are the bulk dual of the field theory deformation (5.118).

In the rest of this subsection we ignore triple trace deformations and focus our attention entirely on the double trace deformations. As explained in Appendix 5.C.1, in this case the modified boundary condition in (5.122) can be undone by a rotation in the space of scalar

[31] This expansion is in conformity with (5.255) because $\zeta = \frac{1}{2}$ for the $m^2 = -2$ scalars of Vasiliev theory.

fields. This is most easily seen in the special case that we have a single S type scalar and a single F type scalar so that both the a and α indices run over a single value and can be ignored. Let us define the rotated fields

$$\frac{S'}{g_a} = \cos\theta \frac{S}{g_a} + \sin\theta \frac{F}{g_\alpha}, \quad \frac{F'}{g_\alpha} = \cos\theta \frac{F}{g_\alpha} - \sin\theta \frac{S}{g_a} \tag{5.124}$$

with

$$\tan\theta = \frac{\pi N \sqrt{h_+^a h_-^\alpha}}{2k} d_{a\alpha}. \tag{5.125}$$

Notice that the field redefinition (5.124) leaves the bulk action invariant. Moreover, it follows from (5.123) that

$$(S')^{(2)} = (F')^{(1)} = 0.$$

In other words the rotated fields S' and F' obey the same bulk equations and same boundary conditions in the presence of the double trace deformation as the unrotated fields S and F obey in their absence.

At first sight this observation leads to the following paradox. A double trace deformation by the parameter d may be thought of as the result of compounding two double trace deformations of magnitude d_1 and d_2 respectively, such that $d_1 + d_2 = d$. As the system after the deformation by d_1 is apparently self similar to the system in its absence, it would appear to follow that the rotation that results from the deformation with $d_1 + d_2$ is simply the sum of the rotations corresponding to d_1 and d_2 respectively; in other words that the rotation angle θ is linear in d. This conclusion is in manifest contradiction with (5.125).

The resolution of this contradiction lies in the fact that the systems with and without the double trace deformations are not, infact, isomorphic. The reason for this is that the boundary counterterm action does not take the simple $\theta = 0$ form in terms of rotated fields in the system with the double trace deformation (see Appendix 5.C.1). In the theory with double trace deformations there is, in particular, a nonzero contact term in the two point functions of the two operators with distinct scaling dimensions; this contact term is absent in the original system.

Spin half fermions

The Vasiliev theories we study include a collection of real fermions ψ_1^a and ψ_2^a propagating in AdS$_4$ space. It is sometimes useful to work with the complex fermions $\psi^a = \frac{\psi_1^a + i\psi_2^a}{\sqrt{2}}$ and $\bar{\psi}^a = \frac{\psi_a^1 - i\bar{\psi}_2^a}{\sqrt{2}}$. Let us suppose that the bulk action takes the form

$$\sum_a \frac{1}{g_a^2} \int \bar{\psi}^a D_\mu \Gamma^\mu \psi_a. \tag{5.126}$$

Using the rules described for instance in [70], the two point function for the operator dual to ψ^a is easily computed and we find the answer

$$\frac{1}{g_a^2} \frac{\vec{x} \cdot \vec{\sigma}}{\pi^2 x^4}. \tag{5.127}$$

The same result also applies to the two point functions of the operators dual to ψ_1^a and ψ_2^a independently.

In analogy with the bosonic case described in the previous subsection, the formula (5.127) presumably applies only with the simplest choice of boundary counterterms [81, 82, 83, 84] - the analogue of $\theta_0 = 0$ in Appendix 5.C.1- consistent with the boundary conditions described in [70]. Though we will not perform the required careful analysis in this paper, it seems likely that the fermionic analogue of Appendix 5.C.1 would find a one parameter set of inequivalent boundary actions that lead to the same boundary conditions. From the bulk viewpoint this ambiguity is likely related to the freedom associated with rotating a bulk spinor ψ_1 into $\Gamma_5\psi_2$ (Γ_5 is the bulk chirality matrix). We ignore this potential complication in the rest of this subsection, and focus on the simple canonical case described in [70].

Let the field theory operator proportional to ψ^a be denoted by Ψ^a. Let us assume that we know from field theory that

$$\langle \Psi^a(x)\bar{\Psi}^b(0)\rangle = \delta^{ab}\frac{h_\psi 2N(\vec{x}\cdot\vec{\sigma})}{(4\pi)^2 x^4}. \tag{5.128}$$

We will now describe the boundary conditions dual to a field theory double trace deformation. Let the fermionic fields have the small z expansion

$$\begin{aligned} \psi_1^a &= z^{\frac{3}{2}}\left(\zeta_{1+}^a + \zeta_{1-}^a\right) + \mathcal{O}(z^{\frac{5}{2}}), \\ \psi_2^a &= z^{\frac{3}{2}}\left(\zeta_{2+}^a + \zeta_{2-}^a\right) + \mathcal{O}(z^{\frac{5}{2}}). \end{aligned} \tag{5.129}$$

Above the subscripts $+$ and $-$ denote the eigenvalue of the corresponding fermions under parity.

Using the procedure of the previous subsection, the bulk dual of the field theory double trace deformation

$$\frac{\pi}{4k}\left[s_{ab}\left(\bar{\Psi}^a+\Psi^a\right)\left(\bar{\Psi}^b+\Psi^b\right) - t_{ab}\left(\bar{\Psi}^a-\Psi^a\right)\left(\bar{\Psi}^b-\Psi^b\right) + u_{ab}\left(\bar{\Psi}^a+\Psi^a\right)\ i\left(\bar{\Psi}^b-\Psi^b\right)\right]$$

is given by the modified boundary conditions

$$\begin{aligned} \frac{\zeta_{1+}^a}{g_a} &= \frac{N\pi\sqrt{h_\psi^a h_\psi^b}}{8k}\left(s_{ab}\frac{\zeta_{1-}^b}{g_b} + \frac{1}{2}u_{ab}\frac{\zeta_{2-}^b}{g_b}\right), \\ \frac{\zeta_{2+}^a}{g_a} &= \frac{N\pi\sqrt{h_\psi^a h_\psi^b}}{8k}\left(t_{ab}\frac{\zeta_{2-}^b}{g_b} + \frac{1}{2}u_{ba}\frac{\zeta_{1-}^b}{g_b}\right). \end{aligned} \tag{5.130}$$

5.5.3 Gauging a global symmetry

As originally introduced by Witten [65], gauging a global symmetry with Chern-Simons term in the boundary CFT is equivalent to changing the boundary condition of the bulk gauge field corresponding to the boundary current of the global symmetry. We will review this relation in this subsection and in Appendix 5.B.

Let us start by considering a boundary CFT with $U(1)$ global symmetry. The current associated to this global symmetry is dual to a $U(1)$ gauge field A_μ in the bulk. In the $A_z = 0$ radial gauge, the action for the gauge field A_μ is

$$\frac{1}{4g^2}\int \frac{d^3\vec{x}dz}{z^4} F_{\mu\nu}F^{\mu\nu} = \int d^3\vec{x}dz \left(\frac{1}{2g^2}\partial_z A_i \partial_z A_i + \frac{1}{4g^2}F_{ij}F_{ij}\right). \tag{5.131}$$

Onshell the bulk action evaluates to

$$\int d^3\vec{x} \left(\frac{1}{2g^2} A_i \partial_z A_i\right). \tag{5.132}$$

where the integral is taken over a surface of constant z for small z. The equations of motion w.r.t. the boundary gauge field impose the *electric* boundary condition

$$\frac{1}{g^2}\partial_z A_i\big|_{z=0} = 0. \tag{5.133}$$

Near $z = 0$, the most general solution to the gauge field equations of motion is

$$A_i = A_i^1(x) + zA_i^2(x).$$

The boundary condition (5.133) forces A_i^2 to vanish but allows $A_i = A_i^1$, the value of the gauge field on the cut off surface, to fluctuate freely at the boundary $z = 0$. The theory so obtained is the conceptual equivalent of the 'alternate' quantized scalar theory described in Appendix 5.C.1.

If we add a boundary $U(1)$ Chern-Simons term to the bulk action [32] (in Euclidean signature)

$$\frac{ik}{4\pi}\int d^3\vec{x}\, \epsilon_{ijk} A_i \partial_j A_k, \tag{5.134}$$

[32]This is the same as adding a term in the bulk action proportional to $\int F \wedge F$ as this term is the total derivative of the Chern Simons term

and allow arbitrary variation δA_i at $z = 0$, the equation of motion of the boundary field A_i generates the modified boundary condition

$$\frac{1}{g^2}\partial_z A_i + \frac{ik}{2\pi}\epsilon_{ijk}\partial_j A_k\big|_{z=0} = 0, \tag{5.135}$$

which is the electric-magnetic mixed boundary condition. By the AdS/CFT dictionary, this is also equivalent to adding the term (5.134) into the boundary theory, where A_i is now interpreted as the three dimensional gauge field coupled to the $U(1)$ current.

This procedure can be straightforwardly generalized to $U(M)$. Adding the $U(M)$ Chern-Simons action on the boundary

$$\frac{ik}{4\pi}\int d^3\vec{x}\epsilon_{ijk}\mathrm{tr}\left(A_i\partial_j A_k + \frac{2}{3}A_i A_j A_k\right). \tag{5.136}$$

modifies the electric boundary condition to

$$\frac{1}{g^2}\partial_z A_i + \frac{ik}{2\pi}\epsilon_{ijk}\left(\partial_j A_k + A_j A_k\right)\big|_{z=0} = 0. \tag{5.137}$$

Note that this mixed boundary condition is still gauge invariant.

Of course $\partial_z A_i$ is determined in terms of A_i by the equations of motion. As the equations of motion are linear, the relation between these quantities is linear - but nonlocal- and takes the form

$$\partial_z A_i(q) = G_{ij}(q)A_j(q).$$

The function $G_{ij}(q)$ has a simple physical interpretation; it is the two point function of the current operator (with natural normalization) in the theory at $k = \infty$ (at this value of k the boundary condition (5.137) is simply the standard Dirichlet boundary condition). A simple computation yields

$$\langle J_i(p)J_j(-q)\rangle = \frac{1}{2g^2}G_{ij}(q)\delta^3(p-q) = -\frac{|p|}{2g^2}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)(2\pi)^3\delta^3(p-q). \tag{5.138}$$

Note that here we have normalized the current coupled to the Chern-Simons gauge field according to the convention for nonabelian gauge group generators, $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$ for generators t^a, t^b in the fundamental representation. This is also the normalization convention we use to define the Chern-Simons level k (which differs by a factor of 2 from the natural convention for $U(1)$ gauge group).

Recall that (5.138) yeilds the two point functions of the 'ungauged' theory - i.e. the theory with $k = \infty$. Our analysis of the dual boundary theory to this ungauged system, we find it convenient to work with currents normalized so that

$$\langle J_i(p) J_j(-q)\rangle = -\frac{\widetilde{N}|p|}{32}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)(2\pi)^3\delta^3(p-q). \tag{5.139}$$

Our convention is such that in the free theory $\widetilde{N}$ counts the total number of complex scalars plus fermions (i.e. the two point function for the charge current for a free complex scalar is equal to that of the free complex fermion and is given by (5.139) upon setting $\widetilde{N} = 1$, see Appendix 5.F). In order that (5.138) and (5.139) match we must identify

$$g^2 = \frac{16}{\widetilde{N}},$$

so that the effective boundary conditions on gauge fields become

$$\frac{\pi\widetilde{N}}{8k}\partial_z A_i + i\epsilon_{ijk}\partial_j A_k\big|_{z=0} = 0. \tag{5.140}$$

In summary, gauging of the global symmetry is affected by the boundary conditions (5.140). Note that the boundary conditions (5.140) constrain only the boundary field strength F_{ij}. Holonomies around noncontractable cycles are unconstrained and must be integrated over.

5.5.4 Deconstruction of boundary conditions: general remarks

The bulk dual of the finite Chern Simons coupling

With essential preliminaries taken care of we now turn to the main topic of this subsection, namely the deconstruction of the supersymmetric boundary conditions of the previous section.

The Vasiliev dual of free susy theories was described in Section 5.2.4. What is the Vasiliev dual to the free field theory deformed *only* by turning on a finite Chern Simons t'Hooft coupling $\lambda = \frac{N}{k}$? The deformation we study is unaccompanied by any potential and Yukawa terms - in particular those needed to preserve supersymmetry - and so is not supersymmetric. Consequently the comparisons between susy Lagrangians and boundary conditions, presented later in this section, does not directly address the question raised here. As we will see, however, the answer to this question is partly constrained by symmetries, and receives indirect inputs from our analysis of susy theories below.

We first recall that it was conjectured in [21] that the bulk dual to turning on λ involves a modification of the *bulk* Vasiliev equations by turning on an appropriate parity violating phase, $\theta(X)$, as a function of λ. The results of the previous section clearly substantiate this conjecture [33]. It is possible, however, that in addition to turning on the phase, a nonzero Chern Simons coupling also results in modified boundary conditions on bulk scalars and fermions. We now proceed to investigate this possibility.

A consideration of symmetries greatly constrains possible modifications of boundary conditions. Recall that the Vasiliev dual to free susy theories possesses a $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$

[33]As those results are valid only for the linearized theory, they unfortunately cannot distinguish between a constant phase and a more complicated phase function; we return to this issue below.

global symmetry. In the dual boundary theory the $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ symmetry rotates the fundamental bosons and fermions respectively, and is preserved by turning on a nonzero Chern Simons coupling. A constant phase in Vasiliev's equations also preserves this symmetry. It follows that all accompanying boundary condition deformations must also preserve this symmetry.

Parity even and odd bulk scalars respectively transform in the (adjoint + singlet, singlet) and (singlet, adjoint+singlet) representations of the $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ symmetry. The only conformally invariant modifications of boundary condition that preserve this symmetry are those dual to the double trace coupling of the parity odd and parity even singlet scalars, and that dual to the triple trace deformation of three parity even singlet scalars.

The conjectures of the previous section strongly constrain the double trace type deformation of boundary conditions induced by the Chern Simons coupling [34]. Let us, for instance, compare Lagrangian and boundary conditions of the fixed line of $\mathcal{N} = 1$ theories described in the previous subsection. The double trace scalar potential in these theories is listed in (5.160) below and vanishes at $\omega = -1$. On the other hand the rotation γ in the scalar boundary conditions for the dual Vasiliev system is listed in (5.101), and vanishes for the dual of $\omega = -1$. In other words the Vasiliev dual to the Chern-Simons theory with no scalar potential obeys boundary conditions such that all 'parity even' scalars continue to have $\Delta = 1$ boundary conditions, while all 'parity odd' scalars continue to have $\Delta = 2$ boundary conditions. While the argument presented above holds only for $n = 2$, the result continues to apply at $n = 4$ and $n = 6$ as well, as we will see in more detail in the detailed

[34] Our analysis of boundary conditions in the previous section was insensitive to triple trace type boundary conditions, and so does not constrain the triple trace type modification.

comparisons below. [35]

We turn now to the fermions. Bulk fermions transform in the (fundamental, antifundamental) and (antifundamental, fundamental) of the free symmetry algebra. There is, of course, a natural double trace type singlet boundary condition deformation with this field content (this deformation has the same effect on boundary conditions as a double trace field theory term $(\phi_a\bar{\psi}^b)(\psi_b\bar{\phi}^a)$ where a and b are global symmetry indices and brackets denote the structure of gauge contractions). Perhaps surprisingly, we will now argue that merely turning on the Chern Simons term *does* induce such a boundary condition deformation. More precisely, it turns out that the bulk theory with trivial boundary conditions on fermions corresponds to a quantum field theory with fermion double trace potential equal to

$$-\frac{6\pi}{k}\bar{\Psi}\Psi$$

for every single trace Fermionic operator.

We present a heuristic argument for this conclusion in Appendix 5.E by comparing the Lagrangian and boundary conditions of the line of $\mathcal{N}=1$ theories with a single chiral multiplet. However the most convincing argument for this conclusion is that it leads to consistent results between the Lagrangian and boundary conditions in *every* case we study in detail later in this section.

[35] For the case $n=4$ consider, for instance, the $\mathcal{N}=2$ theory with two fundamental chiral multiplets. The free theory has a $U(2)\times U(2)$ symmetry. The interacting theory preserves the diagonal $SU(2)$ subgroup of this symmetry (corresponding to rotations of the two chiral multiplets). The parity odd and even single trace operators in this theory each transform in the $1+3$ representations of this symmetry. The allowed double trace deformations of this interacting theory couple the parity even 3 with the parity odd 3 and the parity even scalar with the parity odd scalar. It so happens that these two terms appear with the same coefficient in both the field field theory potential (5.299) and the corresponding Vasiliev boundary conditions (the fact that these terms appear with the same coefficient in (5.95) is simply the fact that the singlet monomial I, appears on the same footing as the triplet monomials $\psi_2\psi_3, \psi_3\psi_4, \psi_4\psi_2$ in the scalar boundary conditions). These facts together demonstrate that the Chern Simons term (which could have acted only on the singlet double trace term and so would have 'split the degeneracy' between singlets and triplets) has no double trace type effect on scalar boundary conditions.

In order to compensate for the shift described above, will find it useful, in our analysis below, to compare Fermionic boundary conditions with a shifted field theory Lagrangian: one in which we add by hand the double trace term $\frac{6\pi}{k}\bar{\Psi}\Psi$ for every single trace fermionic field. Bulk fermionic fields have trivial boundary conditions only when the double trace deformations of the corresponding fermionic operators vanish in the shifted field theory Lagrangian.

Special Points in moduli space for scalars

If we wish to specify the bulk dual for a 3d conformal field theory, it is insufficient to specify the bulk action and the boundary conditions for bulk scalars (see Appendix 5.C.1). In order to specify the correlators of the dual theory we must, in addition, specify the precise nature of the boundary dynamics that gives rise the resultant boundary conditions. Inequivalent boundary dynamics that lead to the same boundary conditions result in distinct correlation functions; in particular to different counterterms in correlators.

Of the set of all boundary actions that lead to a particular boundary condition, one is particularly simple ($\theta_0 = 0$ in Appendix 5.C.1); this choice of boundary counterterms ensures that correlators between dimension one and dimension two operators vanish identically (including contact terms). Let us suppose that the dual of a particular quantum field theory is governed by this simple boundary dynamics. Then the dual of this theory deformed by a scalar double trace deformation cannot, in general, also be governed by the same simple boundary dynamics (see Appendix 5.C.1).

In the moduli space of field theories obtained from one another by double trace deformations, it follows that there is a special point at which boundary scalar dynamics is governed by the simple $\theta_0 = 0$ rule. It certainly seems natural to conjecture that this special theory is governed by a Lagrangian with no double trace terms, i.e. the pure Chern Simons theory described in the previous subsection. As we will explain below, this assumption unfortunately appears to clash with an at least equally natural assumption about the AdS/CFT implementation of the boundary Chern Simons gauging of a global symmetry, as we review below.

Identification of bulk and boundary Chern Simons terms

As we have explained in Section 5.5.3, it is very natural to simply identify the boundary field theoretic Chern Simons term with a Chern Simons term for the boundary value of bulk gauge fields. If we make this assumption then it follows that the boundary conditions for bulk vector uniquely specify its boundary dynamics and the comparison of gauge field structures between the bulk and the boundary establish a map between moduli spaces of field theories and the Vasiliev dual. As we have mentioned in the previous subsubsection, however, the results obtained in this manner clash with those obtained from the 'natural' identification of the specially simple field theory as far as scalar double trace operators are concerned. As we explain, one way out of this conundrum is to abandon the 'natural' assumption of the previous subsection. However we do not propose a definitive resolution to this clash in this paper, leaving this for future work.

In the rest of this section we present a detailed comparison between double trace deformations of the field theory Lagrangian and boundary conditions of the dual Vasiliev theory, for the various theories we study, starting with those theories that allow a nontrivial matching of gauge field terms.

5.5.5 $\mathcal{N} = 3$ fixed line with 1 hypermultiplet

In this section we present a detailed comparison of the Lagrangian 5.D.7 of a fixed line of one hypermultiplet $\mathcal{N} = 3$ theories with boundary conditions (5.110) of its conjectured Vasiliev dual.

Boundary conditions for the vector

As described in the Section 5.5.3, the Chern-Simons gauging of the boundary global current results in modifying the boundary conditions for the dual gauge field in the bulk. The modified boundary condition are given by (5.140) which can also be written as

$$\epsilon_{ijk} F_{jk} = \frac{i\pi \tilde{N}}{4k} F_{zi}. \tag{5.141}$$

The form of boundary conditions for gauge field used in Section 5.4

$$B^{(1)}(\vec{x}, z|Y)\big|_{\mathcal{O}(y^2,\bar{y}^2)} = z^2 \left[e^{i\beta}(yFy) + \Gamma e^{-i\beta}(\bar{y}\overline{F}\bar{y})\right] + \mathcal{O}(z^3) \tag{5.142}$$

are equivalent to

$$\epsilon_{ijk} F_{jk} = 2i \tan(\beta - \theta_0) F_{zi}. \tag{5.143}$$

Comparing (5.141) and (5.143) we get

$$\tan(\beta - \theta_0) = \frac{\pi \tilde{N}}{8k}. \tag{5.144}$$

From (5.110) we have

$$\beta = \theta_0 + (\tilde{\beta} - \theta_0) P_\Gamma,$$

where $\tilde{\beta}$ is the free parameter that parameterizes the fixed line of boundary conditions (5.110). In particular case of vectors proportional to $P_\Gamma \beta = \tilde{\beta}$. Comparing (5.141), (5.143) and (5.144) it follows that

$$\tan(\tilde{\beta} - \theta_0) = \frac{k_1}{k_2} \tan\theta_0, \tag{5.145}$$

where

$$\tan\theta_0 = \frac{\pi \tilde{N}}{8k_1} = \frac{\pi N h_A}{2k_1}. \tag{5.146}$$

Here h_A is the ratio of the two point function of current at the ungauged $\mathcal{N} = 3$ point ($k_2 = \infty$) to the two point function in the free theory. (5.145) establishes a clear map between the parameter $\tilde{\beta}$ that labels boundary conditions in (5.110) and the parameter $\frac{k_1}{k_2}$ that labels the fixed line of dual field theories.

Scalar double trace deformation

In this subsection we compare the scalar double trace operators in the field theory Lagrangian (5.D.7) with the boundary conditions for scalar fields (5.110) in the Vasiliev dual.

The scalar double trace deformation in the Lagrangian (5.D.7) is given by

$$\begin{aligned} V_s &= \frac{2\pi}{k_1}\Phi^a_+\Phi^b_-\eta_{ab} + \frac{2\pi}{k_2}\left(\Phi^0_+\Phi^0_- + \Phi^a_+\Phi^b_-\eta_{ab}\right), \\ &= -\frac{2\pi}{k_1}\Phi^0_+\Phi^0_- + \frac{2\pi}{k_1}\left(1 + \frac{k_1}{k_2}\right)\Phi^i_+\Phi^i_-. \end{aligned} \tag{5.147}$$

This potential interpolates between that of the $\mathcal{N} = 3$ ungauged theory ($k_2 = \infty$) and $\mathcal{N} = 4$ theory ($k_2 = -k_1$). The two point function of $\Phi^a_\pm$ are twice of those given in (5.346) and thus matches with (5.119). The boundary conditions for scalar fields are described by the rotation angle

$$\gamma = \theta_0 P_1 + \tilde{\beta} P_{\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}. \tag{5.148}$$

The double trace term $\frac{2\pi}{k_1}(1 + \frac{k_1}{k_2})\Phi^i_+\Phi^i_-$ couples two $SO(3)$ vectors. The rotation angle that multiples $P_{\psi_1\psi_4,\psi_2\psi_4,\psi_3\psi_4}$ in (5.123) is determined by the coefficient of this term. The precise relationship between these may be obtained as follows. Let us suppose that the formula (5.123) applies starting from some as yet unknown point, $\tilde{\beta} = \tilde{\beta}_0$, in the moduli space of theories. In other words we hypothesize that $\theta_0 = 0$ (in the language of Appendix 5.C.1) for the point in moduli space with $\tilde{\beta} = \tilde{\beta}_0$. Let us also suppose that $k_2 = (k_2)_0$ corresponding field theory. It follows then from (5.123), (5.148) and (5.147) that (see below for the numerical values of the proportionality constants)

$$\tan(\tilde{\beta} - \tilde{\beta}_0) \propto \frac{1}{k_2} - \frac{1}{(k_2)_0}.$$

Case: $\tilde{\beta}_0 = 0$:

Purely from the viewpoint of the scalars it is natural to conjecture that $\tilde{\beta}_0 = 0$ and $(k_2)_0 = -k_1$. This conjecture is motivated by the following observations. The contact term in the two point function between Φ^i_+ and Φ^i_- vanishes in the field theory dual to bulk boundary conditions governed by the parameter $\tilde{\beta}_0$. At leading order in boundary perturbation theory (i.e. at order $1/k$) a naive computation yields a contact term proportional to the double trace coupling of Φ^i_+ and Φ^i_-. Thus appears to imply that the special field theory have a vanishing double trace term; this occurs at the $\mathcal{N} = 4$ point and so $\tilde{\beta}_0 = 0$. If we make this assumption it then follows that that

$$\tan\tilde{\beta} = \tan\theta_0 \left(1 + \frac{k_1}{k_2}\right), \quad \text{with} \quad \tan\theta_0 = \frac{N\pi}{2k_1}\sqrt{h_+h_-}, \tag{5.149}$$

where h_+ and h_- is the ratio of two point function for Φ_+ and Φ_- respectively in the interacting ($\mathcal{N} = 4$ point) to free theory. Unfortunately (5.149) conflicts with (5.145), so both relations cannot be simultaneously correct.

Case: $\tilde{\beta}_0 = \theta_0$:

The conflict with (5.151) vanishes if we instead assume that

$$\tilde{\beta}_0 = \theta_0. \tag{5.150}$$

This is dual to the 'ungauged' $\mathcal{N} = 3$ theory and so it follows that and $(k_2)_0 = \infty$. Under this assumption it follows that

$$\tan(\tilde{\beta} - \theta_0) = \tan\theta_0 \left(\frac{k_1}{k_2}\right), \quad \text{with} \quad \tan\theta_0 = \frac{N\pi}{2k_1}\sqrt{h_+ h_-}, \tag{5.151}$$

where h_+ and h_- is the ratio of two point function for Φ_+ and Φ_- respectively in the interacting ('ungauged' $\mathcal{N} = 3$ point) to free theory. Note that (5.151) perfectly matches (5.146) if $h_A = \sqrt{h_+ h_-}$. It is plausible that supersymmetry enforces this relationship on field theory operators, but we will not attempt to independently verify this relationship in this paper.

Perhaps the simplest resolution of the clash betwen (5.149) and (5.145) is obtained by setting $\tilde{\beta}_0 = \theta_0$. Before accepting this suggestion we must understand why the contact term in the scalar- scalar two point function vanishes at the $\mathcal{N} = 3$ rather than at the $\mathcal{N} = 4$ point (where the double trace term in the Lagrangian vanishes). As discussions relating to contact terms are famously full of pitfalls; we postpone the detailed study of this question to later work.

Coefficient of the scalar double trace deformation

The double trace term in (5.147) that couples two $SO(3)$ scalars is $\frac{2\pi}{k_1}\Phi_+^0\Phi_-^0$. Note that the coefficient of this term is independent of k_2, which matches with the fact that the coefficient of P_1 in (5.148) is independent of $\tilde{\beta}$.

If we assume that $\tilde{\beta}_0 = \theta_0$ for this term as well we once again find the second of (5.149), where h_+ and h_- have the same meaning as in (5.149), except that the two point function in question is that of the scalar operator ϕ^0. We conclude that ϕ^a and ϕ^0 have equal values of h_+h_-. If, instead, $\tilde{\beta}=0$ then a very similar equation holds; the only difference is that h_+h_- would then compute ratios of the interacting and free two point functions at the $\mathcal{N}=4$ point.

Fermionic double trace deformation

The fermionic double trace deformation for this fixed line is given by

$$V_3=\frac{2\pi}{k_1}\left(\frac{1}{2}\bar{\Psi}^a\Psi^b\delta^{ab}-2\bar{\Psi}^0\Psi^0-\bar{\Psi}^0\bar{\Psi}^0-\Psi^0\Psi^0\right)+\frac{2\pi}{k_2}\left(\bar{\Psi}^a\Psi^b\eta^{ab}+\frac{1}{2}\bar{\Psi}^a\bar{\Psi}^b\eta_{ab}+\frac{1}{2}\Psi^a\Psi^b\eta^{ab}\right). \tag{5.152}$$

Adding $\delta V_f=\frac{3\pi}{k}\bar{\psi}^a\psi^a$ in order to account the effect of finite Chern Simons level as described earlier, we obtain the shifted potential

$$V_3+\delta V_f=-\frac{\pi}{k_1}(\Psi^a-\bar{\Psi}^a)(\Psi^b-\bar{\Psi}^b)\delta^{ab}+\frac{\pi}{k_1}\left(1+\frac{k_1}{k_2}\right)\left(\bar{\Psi}^a+\Psi^a\right)\eta_{ab}\left(\bar{\Psi}^b+\Psi^b\right). \tag{5.153}$$

The two point function of $\langle\bar{\Psi}^a\Psi^b\rangle$ is twice of the that given in (5.346) because Ψ^a are constructed out of field doublets and thus matches with (5.128).

The rest of the analysis closely mimics the study of scalar double trace deformations presented in the previous subsection. We associate(in the boundary conditions) the projector P^a_ψ with the real Lagrangian deformation $[i(\psi^a-\bar{\psi}^a)]^2$ and $P_{\Gamma\psi^a}$ with the other real Lagrangian deformation $(\psi^a+\bar{\psi}^a)^2$. As for the scalar double trace deformations, (5.130) yields results consistent with (5.145) if and only if we assume that (5.130) applies for deformations about the special point $\tilde{\beta}=\theta_0$. Given this assumption (5.110) and (5.130) matches with the identification (5.151) with $\sqrt{h_+h_-}=h_\psi$ and h_ψ interpreted as the ratio of $\langle\bar{\Psi}^a\Psi^b\rangle$ at $\mathcal{N}=3$ point to the free theory.[36]

[36]If, on the other hand, (5.130) had applied for deformations around $\tilde{\beta}=0$ we would instead have found

5.5.6 $\mathcal{N} = 3$ fixed line with 2 hypermultiplets

In this section we compare the Lagrangian for the fixed line of two hypermultiplet theories presented in (5.D.9) with the boundary conditions (5.115) of the conjectured Vasiliev duals. The field theories under study interpolate between the ungauged $\mathcal{N} = 3$ theory ($k_2 = \infty$) and the $\mathcal{N} = 6$ theory (at $k_2 = -k_1$).

Vector field boundary conditions

The comparison here is very similar to that performed in the previous subsection, and our presentation will be brief. Making the natural assumptions spelt out in the previous section, the gauge field boundary conditions listed in (5.115) assert that

$$\beta = \theta_0 + (\tilde{\beta} - \theta_0)P_\Gamma.$$

Using (5.144) we find

$$\tan(\tilde{\beta} - \theta_0) = \frac{k_1}{k_2} \tan 2\theta_0. \tag{5.154}$$

with the identification

$$\tan(2\theta_0) = \frac{\pi \tilde{N}}{8k_1} = \frac{\pi N h_A}{k_1}$$

where h_A is interpreted as the ratio of the two point function of the flavor current in the ungauged $\mathcal{N} = 3$ theory to the free theory.

agreement with (5.149) with $\sqrt{h_+ h_-} = h_\psi$, where h_ψ would have been interpreted as the ratio of $\langle \bar{\Psi}^a \Psi^b \rangle$ at $\mathcal{N} = 4$. Of course these results contradict (5.145).

Scalar double trace deformation

The scalar double trace deformation for this case, in the notation defined in Appendix 5.D.9, is given by

$$\begin{aligned} V_s &= \frac{\pi}{k_1}\Phi_+^{Ii}\Phi_-^{Jj}\eta^{IJ}\eta_{ij} - \frac{2\pi}{k_2}\Phi_+^{I0}\Phi_-^{J0}\eta^{IJ} \\ &= \frac{\pi}{k_1}\left(\Phi_+^{Ii}\Phi_-^{Jj}\eta^{IJ}\eta_{ij} + 2\Phi_+^{I0}\Phi_-^{J0}\eta^{IJ}\right) - \frac{2\pi}{k_1}\left(1+\frac{k_1}{k_2}\right)\Phi_+^{I0}\Phi_-^{J0}\eta^{IJ}. \end{aligned} \tag{5.155}$$

Due the fact that Φ_+^{Ii} and $\bar{\Phi}_-^{Ii}$ are made of two field doublets, there free two point function are four times of those given in (5.346) and thus twice of those given in (5.119). The boundary conditions of the dual scalars listed in (5.D.9) is governed by

$$\gamma = \theta_0 P_{1,\psi_i\psi_a,\psi_a\psi_b} - \tilde{\beta}P_{\psi_i\psi_j}, \qquad P_{1,\psi_i\psi_j,\psi_i\psi_a,\psi_a\psi_b}\tilde{f}_{1,2} = \tilde{f}_{1,2}. \tag{5.156}$$

As in the previous section the coefficient of the double trace deformations (5.155) and the boundary conditions of scalars in (5.156) are both respectively independent of k_2 and $\tilde{\beta}$ in every symmetry channel but one (i.e. (vector, scalar) under $SU(2)\times SU(2)$). Comparing coefficients in this special channel we find that (5.156) and (5.D.9) agree with (5.144)if and only if we assume that (5.123) applies for deformations of $\tilde{\beta}$ away from the special point $\tilde{\beta}_0 = \theta$ at which point $k_2 = \infty$.

$$\tan(\tilde{\beta}-\theta_0) = \tan 2\theta_0\left(\frac{k_1}{k_2}\right) \qquad \text{with} \qquad \tan 2\theta_0 = \frac{\pi N}{k_1}\sqrt{h_+h_-}, \tag{5.157}$$

with $h_\pm$ interpreted as the ratio of two point function in $\mathcal{N}=3$ ungauged point to free theory.

On the other hand upon assuming $\tilde{\beta}_0 = 0$ we find

$$\tan(\tilde{\beta}+\theta_0) = \tan 2\theta_0\left(1+\frac{k_1}{k_2}\right) \qquad \text{with} \qquad \tan 2\theta_0 = \frac{\pi N}{k_1}\sqrt{h_+h_-}, \tag{5.158}$$

with $h_\pm$ interpreted as the ratio of two point function in $\mathcal{N}=6$ point to free theory. This is in contradiction with (5.154).

We now turn to the comparison of the double trace terms and boundary conditions in all other channels (i.e. (scalar, scalar), (vector, vector) and (scalar, vector) under $SO(3)\times SO(3)$. In each case if we assume that (5.123) applies starting from the special point $\tilde{\beta}_0=\theta_0$, we find the second of (5.157) with with $h_\pm$ interpreted as the ratio of two point function in $\mathcal{N}=3$ ungauged point to free theory for the appropriate scalar. This suggests that the product h_+h_- is the same for scalars in all four symmetry channels; this product is also equal to h_A^2. It is possible that this equality is consequence of $\mathcal{N}=3$ supersymmetry of the field theory; we leave the verification of this suggestion to future work.

Fermionic double trace deformation

The fermionic double trace deformation for this case, in the notation defined in Appendix 5.D.9, after compensating by a for the chern simons shift [37], is given by

$$\begin{aligned}
V_f+\delta V_f=&\frac{\pi}{k_1}\left(\bar{\Psi}^{Ii}\Psi^{Jj}\delta^{IJ}\delta^{ij}+\bar{\Psi}^{Ii}\Psi^{Jj}\eta^{IJ}\delta^{ij}+\left(\bar{\Psi}^{0i}\bar{\Psi}^{0j}\eta_{ij}+\Psi^{0i}\Psi^{0j}\eta_{ij}\right)\right)\\
&+\frac{\pi}{k_2}(\bar{\Psi}^{I0}+\Psi^{I0})(\bar{\Psi}^{J0}+\Psi^{J0})\eta_{IJ}.\\
&=\frac{\pi}{k_1}\left(\bar{\Psi}^{Ii}\Psi^{Jj}\delta^{IJ}\delta^{ij}+\bar{\Psi}^{Ii}\Psi^{Jj}\eta^{IJ}\delta^{ij}+\left(\bar{\Psi}^{0i}\bar{\Psi}^{0j}\eta_{ij}+\Psi^{0i}\Psi^{0j}\eta_{ij}\right)\right.\\
&\left.-(\bar{\Psi}^{I0}+\Psi^{I0})(\bar{\Psi}^{J0}+\Psi^{J0})\eta_{IJ}\right)+\frac{\pi}{k_1}\left(1+\frac{k_1}{k_2}\right)(\bar{\Psi}^{I0}+\Psi^{I0})(\bar{\Psi}^{J0}+\Psi^{J0})\eta_{IJ}.
\end{aligned} \tag{5.159}$$

The two point function $\langle\bar{\Psi}^{Ii}\Psi^{Jj}\rangle$ is twice of that given by (5.128).

[37]The compensating factor in this case is $\delta V_f=\frac{3\pi}{2k_1}\bar{\Psi}^{Ii}\Psi^{Ii}$

The bulk boundary conditions are generated by

$$\alpha = \theta_0(P_{\psi_i,\psi_a} + P_{\psi_i\psi_j\psi_a,\psi_i\psi_a\psi_b,\psi_4\psi_5\psi_6} - P_{\psi_a\Gamma}) + \tilde{\beta}(P_{\psi_i\Gamma} - P_{\psi_1\psi_2\psi_3}).$$

Consistency requires us to assume that (5.130) applies for deviations away from $\tilde{\beta} = 0$ (i.e. from the ungauged $\mathcal{N} = 3$ theory). Applying (5.130) we recover (5.157) provided $h_\psi = \sqrt{h_+h_-}$ where h_ψ is the ratio the two point function $\langle\bar{\Psi}^{Ii}\Psi^{Jj}\rangle$ at the ungauged $\mathcal{N} = 3$ point to free theory.[38]

5.5.7 Fixed Line of $\mathcal{N} = 1$ theories

We now turn to the comparison of the Lagrangian (5.300) of the large N fixed line of $\mathcal{N} = 1$ field theories with the boundary conditions (5.98) (a beta function is generated at finite N, the zeros of this beta function are the two ends of the line we study below). We restrict attention to the case $M = 1$. The field content of the theory is a single complex scalar ϕ together with a single complex fermion ψ.

Scalar Double trace terms

The (scalar)(scalar) double trace potential in (5.300) is given by

$$\frac{2\pi(1+\omega)}{k}\bar{\phi}\phi\bar{\psi}\psi. \tag{5.160}$$

$\omega = -1$ is the $\mathcal{N} = 1$ theory with no superpotential while $\omega = 1$ is the $\mathcal{N} = 2$ theory. The two point functions of the constituent single trace operators, $\bar{\phi}\phi$ and $\bar{\psi}\psi$, are given, in the free theory, by (5.346) (note that this corresponds to $h_+ = h_- = \frac{1}{2}$ in (5.119)).

[38]If, instead, (5.130) had applied starting from $\tilde{\beta} = 0$ we would have found consistency with (5.158) provided $h_\psi = \sqrt{h_+h_-}$ where h_ψ interpreted as the ratio the two point function $\langle\bar{\Psi}^{Ii}\Psi^{Jj}\rangle$ at $\mathcal{N} = 6$ point to free theory. This result contradicts the gauge field matching and so cannot apparently cannot be correct.

The $n = 2$ Vasiliev dual to this system is conjectured to have boundary conditions listed in (5.101). Specifically the boundary conditions require B to take the form

$$B(x,z) = zf_1(x)\left((1+\Gamma)\cos\gamma + i(1-\Gamma)\sin\gamma\right) + if_2(x)z^2\left((1-\Gamma)\cos\gamma + i(1+\Gamma)\sin\gamma\right) \tag{5.161}$$

where f_1 and f_2 are real constants and γ ranges from zero (for the $\mathcal{N} = 1$ theory with no superpotential) to $\gamma = \theta_0$ (for the $\mathcal{N} = 2$ theory). Notice that the shift change in phase between these two points is θ_0, while the change in the coefficient of the corresponding double trace term in the Lagrangian (5.160) is $\frac{4\pi}{k}$.

In order to establish a map between the Lagrangian parameter ω and the boundary condition parameter γ we need to know the location of the special point, γ_0, in γ parameter space from which (5.123) applies (this is the point with $\theta_0 = 0$ in the language of Appendix 5.C.1). Unlike the previous subsections, in this case we have no information from the gauge field boundary conditions, so the best we can do is to make a guess. We consider two cases.

Case $\gamma_0 = \theta_0$:

The results of the previous subsection suggest that $\gamma_0 = \theta_0$ so that the special point in the moduli space of Vasiliev theories is the $\mathcal{N} = 2$ theory. If this is the case then

$$\tan(\theta_0 - \gamma) = \tan\theta_0 \frac{1-\omega}{2}$$

where

$$\tan\theta_0 = \frac{\pi\lambda\sqrt{h_+ h_-}}{2} \tag{5.162}$$

and h_+ gives the ratio of the interacting and free two point functions of $\bar{\phi}\phi$ for the $\mathcal{N} = 2$ theory.

Case $\gamma_0 = 0$:

Purely from the point of view of the scalar part of the Lagrangian, the most natural assumption is $\gamma_0 = 0$ in which case

$$\tan\gamma = \tan\theta_0 \frac{1+\omega}{2}$$

where

$$\tan\theta_0 = \frac{\pi\lambda\sqrt{h_+ h_-}}{2} \tag{5.163}$$

and h_+ gives the ratio of the interacting and free two point functions of $\bar{\phi}\phi$ for the $\mathcal{N} = 1$ theory with no superpotential.

Fermion double trace terms

The (fermion)(fermion) double trace potential term after accounting for the shift described in

$$\begin{aligned} V_f + \delta V_f &= V_f + \frac{6\pi}{k}\bar{\psi}\phi\bar{\phi}\psi \\ &= \frac{\pi(\omega+1)}{k}(\bar{\psi}\phi + \bar{\phi}\psi)^2 - \frac{2\pi}{k}(\bar{\psi}\phi - \bar{\phi}\psi)^2. \end{aligned} \tag{5.164}$$

Here $\omega = -1$ corresponds to the undeformed $\mathcal{N} = 1$ theory and $\omega = 1$ corresponds to the $\mathcal{N} = 2$ theory. The two point function of the operator $\bar{\psi}\phi$ and $\bar{\phi}\psi$ are given in (5.346). Note that this corresponds to $h_\psi = \frac{1}{2}$ in (5.119). The boundary condition for fermions are given by (5.235) with

$$\alpha = \theta_0 P_{\psi_2} + \gamma P_{\psi_1}.$$

As explained in the previous section, the coefficient of the P_{ψ_2} in the boundary conditions is associated with the coefficient of double trace deformation $(i(\bar{\psi}\phi - \bar{\phi}\psi))^2$ while the coefficient of P_{ψ_1} is associated with the double trace deformation $(\bar{\psi}\phi + \bar{\phi}\psi)^2$. Note that this matches with the fact that coefficient of the former are constant along the line while those of the later change along the fixed line.

Using the analysis of Section 5.5.2 we can get a more quantitative match. As in the previous subsubsection it is natural to assume - and we conjecture - that If (5.130) applies starting from the $\mathcal{N}=2$ point, at which the first term in (5.164) has coefficient $\frac{2\pi}{k}$. With this assumption

$$\tan(\theta_0-\gamma)=\tan\theta_0\frac{1-\omega}{2}, \qquad \text{with} \qquad \tan\theta_0=\frac{\pi\lambda h_\psi}{2}, \tag{5.165}$$

where h_ψ is the ratio of interacting to free two point function $\langle\bar{\psi}\phi\ \bar{\phi}\psi\rangle$ in $\mathcal{N}=2$ theory.

If, on the other hand (5.130) were to apply starting from the pure $\mathcal{N}=1$ point we would find

$$\tan\gamma=\tan\theta_0\frac{1+\omega}{2}, \qquad \text{with} \qquad \tan\theta_0=\frac{\pi\lambda h_\psi}{2}. \tag{5.166}$$

where h_ψ is the ratio of interacting and free two point function $\langle\bar{\psi}\phi\ \bar{\phi}\psi\rangle$ in $\mathcal{N}=1$ theory with no superpotential. The results of the previous two subsections appear to disfavor this possibility over the one presented in the previous paragraph.

5.5.8 $\mathcal{N}=2$ theory with 2 chiral multiplets

In the final subsection of this section we turn to the comparison of the Lagrangian (5.D.1) (with $M=2$) of the $\mathcal{N}=2$ theory with 2 fundamental chiral multiplets with the boundary conditions (5.95). The theory we study admits no marginal superpotential deformations, and so appears as a fixed point rather than a fixed line at any given value of k_1.

Scalar double trace deformation

The scalar double trace deformation in (5.D.1) is given by

$$V_s=\frac{2\pi}{k}\Phi_+^a\Phi_-^a, \tag{5.167}$$

where $\Phi^a_+ = \bar{\phi}^i \phi_j (\sigma^a)^j{}_i$, $\Phi^a_- = \bar{\psi}^i \psi_j (\sigma^a)^j{}_i$ and a runs over 0,1,2,3. In Appendix 5.F we have computed the two point functions of the operators Φ^a_+ and Φ^a_- in free field theory; the result is given by (5.346) with an extra factor of two to account for the fact that the operators $\Phi^a_\pm$ are constructed out of field doublets. In other words the two point functions of $\Phi^a_\pm$ exactly agree with those presented in (5.119) with h_+ and h_- interpreted as the ratio of the two point functions of $\Phi_\pm$ in the interacting theory and the free theory [39]. With this interpretation (5.123) predicts the boundary conditions of the bulk scalars with $d_{a\alpha} = 1$ (both for the singlet of $SU(2)$ as well as the triplet). Comparing these equations with the actual boundary conditions

$$\gamma = \theta_0, \qquad P_{1,\psi_2\psi_3,\psi_2\psi_4,\psi_3\psi_4} \tilde{f}_{1,2} = \tilde{f}_{1,2},$$

we conclude that $g_a = g_\alpha$ both for singlet scalars as well as for $SU(2)$ triplet scalars.

In order to make a quantitative comparison between the Lagrangian and boundary conditions we need to make an assumption about which point in the moduli space of double trace deformations (5.123) applies from. Given the results of the previous subsections it is natural to guess that (5.123) applies for double trace deformations away from the $\mathcal{N} = 2$ theory. Assuming that the theory with no double trace deformation has trivial scalar boundary conditions, we conclude that

$$\tan\theta_0 = \frac{\pi\lambda\sqrt{h_+ h_-}}{2}. \tag{5.168}$$

where $h_\pm$ are the ratios of two point functions of the scalar operators in the $\mathcal{N} = 2$ and free theories. This equation must hold separately for singlet as well as SU(2) vector sector. It

[39]Here it is ambiguous what is the interacting theory i.e. what is the value of k in theory without the double trace deformations, from where (5.123) applies

seems very likely that $h_+ = h_- = h_s$ for all scalars in which case

$$\tan\theta_0 = \frac{\pi\lambda h_s}{2}. \tag{5.169}$$

fermion double trace deformation

The fermion double trace deformation in this case is given by

$$V_f = \frac{\pi}{k}\bar{\Psi}^a\Psi^a, \tag{5.170}$$

where $\Psi^a = \bar{\phi}^i\psi_j(\sigma^a)^j{}_i$, $\bar{\Psi}^a = \bar{\psi}^i\phi_j(\sigma^a)^j{}_i$ and a runs over 0,1,2,3. In order to compare this double trace potential with boundary conditions, however, we must remove the effect of the Chern Simons term. In other words we should expect the fermion boundary conditions to match with an effective fermion double trace potential given by

$$\delta S = \frac{4\pi}{k}\bar{\Psi}^a\Psi^a.$$

(it is easily verified that a shift by $-\frac{3\pi}{k}$ in the coefficient of $\bar{\Psi}^a\Psi^a$ is equivalent to a shift of $-\frac{6\pi}{k}$ in the coefficient of each fermion). The two point functions of these fields is given by (see Appendix 5.F)

$$\left\langle \Psi^a(x)\bar{\Psi}^b(0)\right\rangle = \frac{N\delta^{ab}h_\psi}{8\pi^2}\frac{\vec{x}\cdot\vec{\sigma}}{x^4},$$

where h_ψ is the ratio of the two point function in the interacting and free theories.

This matches onto the analysis leading up to (5.130) if we set $s = t = 4$ and $u = 0$. Here we assume that (5.130) applies for deformations about the $\mathcal{N} = 2$ point. In this application of (5.130) all factors of g_a relate to fields that are related by $SO(4)$ invariance, and so must be equal. Consequently factors of g_a cancel from that equation. Comparing (5.130) with $s = t = 4$ and $u = 0$ with the actual fermion boundary conditions, in this case

$$\alpha = \theta_0,$$

we recover the equation

$$\tan\theta_0 = \frac{\pi\lambda h_\psi}{2}. \tag{5.171}$$

We see that (5.171) is consistent with (5.168) provided $h_\psi = \sqrt{h_+ h_-}$, with h_ψ interpreted as the ratio of the two point function in the $\mathcal{N}=2$ and free theories. It seems very likely to us that in fact $h_\psi = h_+ = h_- = h_s$.

5.6 The ABJ triality

Having established the supersymmetric Vasiliev theories with various boundary conditions dual to Chern-Simons vector models, we will now use the relation between deformations of the boundary conditions and double trace deformations in the boundary conformal field theory to extract some nontrivial mapping of parameters. In the case of $\mathcal{N}=6$ theory, the triality between ABJ vector model, Vasiliev theory, and type IIA string theory suggests a bulk-bulk duality between Vasiliev theory and type IIA string field theory. We will see that the parity breaking phase θ_0 of Vasiliev theory can be identified with the flux of flat Kalb-Ramond B-field in the string theory.

5.6.1 From $\mathcal{N}=3$ to $\mathcal{N}=4$ Chern-Simons vector models

Let us consider the $\mathcal{N}=3$ $U(N)_k$ Chern-Simons vector model with one hypermultiplets. Upon gauging the diagonal $U(1)$ flavor symmetry with another Chern-Simons gauge field at level $-k$, one obtains the $\mathcal{N}=4$ $U(N)_k \times U(1)_{-k}$ theory. In Section 5.5.5, by comparing the boundary conditions, we have found the relation

$$\tan\theta_0 = \frac{\pi\tilde{N}}{8k} = \frac{\pi\lambda h_A}{2}. \tag{5.172}$$

By comparing the structure of three-point functions with the general results of [51], we see that $\tan\theta_0$ is identified with $\widetilde{\lambda}$ of [51]. Therefore, by consideration of supersymmetry breaking by AdS boundary conditions, we determine the relation between the parity breaking phase θ_0 of Vasiliev theory and the Chern-Simons level of the dual $\mathcal{N}=3$ or $\mathcal{N}=4$ vector model to be

$$\widetilde{\lambda} = \frac{\pi\widetilde{N}}{8k}. \tag{5.173}$$

Recall that $\widetilde{N}$ is defined as the coefficient of the two-point function of the $U(1)$ flavor current J_i in the $\mathcal{N}=3$ Chern-Simons vector model, normalized so that $\widetilde{N}$ is 4 for each *free* hypermultiplet. In notation similar to that of the previous section $\tilde{N} = 4Nh_A$ where h_A is the ratio of the two point function of the flavour currents in the interacting and free theory. Consequently (5.173) may be rewritten as

$$\widetilde{\lambda} = \frac{\pi\lambda h_A}{2}. \tag{5.174}$$

After gauging this current with $U(1)$ Chern-Simons gauge field $\widetilde{A}_\mu$ at level $-k$, passing to the $\mathcal{N}=4$ theory, the new $U(1)$ current which may be written as $J_{new} = -k * d\widetilde{A}$ has a different two-point function than J_i, as can be seen from Section 5.3.1. The two-point function of J_{new} also contains a parity odd contact term, as was pointed out in [65].

We would also like to determine the relation between θ_0 and $\lambda = N/k$, which cannot be fixed directly by the consideration of supersymmetry breaking by boundary conditions. The two-loop result of [21] on the parity odd contribution to the three-point functions also applies to correlators of singlet currents made out of fermion bilinears in supersymmetric

Chern-Simons vector models, since the double trace and triple terms do not contribute to the parity odd terms in the three-point function at this order. From this we learn that $\theta_0 = \frac{\pi}{2}\lambda + \mathcal{O}(\lambda^3)$. Parity symmetry would be restored if we also send $\theta(X) \to -\theta(X)$ under parity, and in particular $\theta_0 \to -\theta_0$. Further, in the supersymmetric Vasiliev theory, θ_0 should be regarded as a periodically valued parameter, with periodicity $\pi/2$. This is because the shift $\theta_0 \to \theta_0 + \frac{\pi}{2}$ can be removed by the field redefinition $\mathcal{A} \to \psi_1 \mathcal{A} \psi_1$, $B \to -i\psi_1 B \psi_1$, where ψ_1 is any one of the Grassmannian auxiliary variables. Note that the factor of i in the transformation of the master field B is required to preserve the reality condition. Essentially, $\theta_0 \to \theta_0 + \frac{\pi}{2}$ amounts to exchanging bosonic and fermionic fields in the bulk.

Giveon-Kutasov duality [85] states that the $\mathcal{N} = 2$ $U(N)_k$ Chern-Simons theory with N_f fundamental and N_f anti-fundamental chiral multiplets is equivalent to the IR fixed point of the $\mathcal{N} = 2$ $U(N_f + k - N)_k$ theory with the same number of fundamental and anti-fundamental chiral multiplets, together with N_f^2 mesons in the adjoint of the $U(N_f)$ flavor group, and a cubic superpotential coupling the mesons to the fundamental and anti-fundamental superfields. Specializing to the case $N_f = 1$ (or small compared to N, k), this duality relates the "electric" theory: $\mathcal{N} = 2$ $U(N)_k$ Chern-Simons vector model with N_f pairs of $\square, \overline{\square}$ chiral multiplets at large N, to the "magnetic" theory obtained by replacing $\lambda \to 1 - \lambda$ and rescaling the value of N, together with turning on a set of double trace deformations and flowing to the critical point. In the holographic dual of this vector model, the double trace deformation in the definition of the magnetic theory simply amounts to changing the boundary condition on a set of bulk scalars and fermions. This indicates that the bulk theory with parity breaking phase $\theta_0(\lambda)$ should be equivalent to the theory with phase $\theta_0(1-\lambda)$, suggesting that the identification

$$\theta_0 = \frac{\pi}{2}\lambda \tag{5.175}$$

is in fact exact in the duality between Vasiliev theory and $\mathcal{N}=2$ Chern-Simons vector models of the Giveon-Kutasov type. By turning on a further superpotential deformation, this identification can be extended to the $\mathcal{N}=3$ theory as well. Together with (5.174), (5.175) then implies that relation $\tan(\frac{\pi}{2}\lambda) = \frac{\pi\widetilde{N}}{8k} = \frac{\pi\lambda h_A}{2}$ in the $\mathcal{N}=3$ Chern-Simons vector model in the planar limit. Note that in the $k\to\infty$ limit where the theory becomes free, this relation becomes the simply $\widetilde{N}=4N$, which follows from our normalization convention of the spin-1 flavor current.

A similar comparison between double trace deformations of scalar operators and the change of scalar boundary condition in the bulk Vasiliev theory lead to the same identification between θ_0 and $\widetilde{N}$, k. Note that in the supersymmetric Chern-Simons vector model, $\widetilde{N}$ by our definition is the two-point function coefficient of a flavor current, which is related to the two-point function coefficient of gauge invariant scalar operators by supersymmetry. However, our $\widetilde{N}$ is a priori normalized *differently* from that of Maldacena and Zhiboedov [51], where $\widetilde{N}$ was defined as the coefficient of two-point function of higher spin currents, normalized by the corresponding higher spin charges.[40]

A high nontrivial check would be to prove the relations (5.174) and (5.175) directly in the field theory using the Schwinger-Dyson equations considered in [21]. In the case of Chern-Simons-scalar vector model, this computation is performed in [66]. It is found in [66] that the relation $\theta_0 = \pi\lambda/2$ holds, whereas the scalar two-point function is precisely proportional to

[40]We thank Ofer Aharony for discussions on this point.

$k \tan\theta_0$ up to a numerical factor that depends on the number of matter fields,[41] remarkably coinciding with our finding in the supersymmetric theory by consideration of boundary conditions and holography. We leave it to future work to establish these relations in the supersymmetric theory using purely large N field theoretic technique.

5.6.2 ABJ theory and a triality

Now let us consider the $\mathcal{N} = 3$ $U(N)_k$ Chern-Simons vector model with two hypermultiplets. Upon gauging the diagonal $U(1)$ flavor symmetry with another Chern-Simons gauge field at level $-k$, one obtains the $\mathcal{N} = 6$ $U(N)_k \times U(1)_{-k}$ ABJ theory. By comparing the boundary conditions, in Section 5.5.6, we have found the formula

$$\tan(2\theta_0) = \frac{\pi\widetilde{N}}{8k} = \pi\lambda h_A, \tag{5.176}$$

where $\widetilde{N}$ is the coefficient of the two-point function of the $U(1)$ flavor current in the $\mathcal{N} = 6$ theory, and h_A, as usual, is the ratio of the flavor current two point function in the interacting and free theory. Note that the factor of 2 in the argument of $\tan(2\theta_0)$ is precisely consistent with the fact that in the $k \to \infty$ limit, the $U(1)$ flavor current which is made out of twice as the $\mathcal{N} = 2$ theory of one hypermultiplet considered in the previous subsection, so that $\widetilde{N}$ is enhanced by a factor of 2 (namely, $\widetilde{N} = 8N$ in the free limit).

Now we can complete our dictionary of "ABJ triality". We propose that the $U(N)_k \times U(M)_{-k}$ ABJ theory, in the limit of large N, k and fixed M, is dual to the $n = 6$ extended supersymmetric Vasiliev theory with $U(M)$ Chan-Paton factors, parity breaking phase θ_0 that is identified with $\frac{\pi}{2}\lambda$, and the $\mathcal{N} = 6$ boundary condition described in Section 5.4.9.

[41][66] adopted the natural field theory normalization for the scalar operator, which would agree with our normalization for the flavor current, and differ from the normalization of [51] by a factor $\cos^2\theta_0$.

The *bulk* 't Hooft coupling can be identified as $\lambda_{bulk} \sim M/N$. In the strong coupling regime where $\lambda_{bulk} \sim \mathcal{O}(1)$, we expect a set of bound states of higher spin particles to turn into single closed string states in type IIA string theory in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ with flat Kalb-Ramond B_{NS}-field flux

$$\frac{1}{2\pi\alpha'} \int_{\mathbb{CP}^1} B_{NS} = \frac{N-M}{k} + \frac{1}{2}. \tag{5.177}$$

In the limit $N \gg M$, we have the identification

$$\theta_0 = \frac{\pi}{2}\lambda = \frac{1}{4\alpha'} \int_{\mathbb{CP}^1} B_{NS} - \frac{\pi}{4}. \tag{5.178}$$

Note that this is consistent with $B_{NS} \to -B_{NS}$ under parity transformation. This suggests that the RHS of Vasiliev's equation of motion involving the B-master field should be related to worldsheet instanton corrections in string theory (in the suitable small radius/tensionless limit).

5.6.3 Vasiliev theory and open-closed string field theory

A direct way to engineer $\mathcal{N}=3$ Chern-Simons vector model in string theory was proposed in [67]. Starting with the $U(N)_k \times U(M)_{-k}$ ABJ theory, one adds N_f fundamental $\mathcal{N}=3$ hypermultiplets of the $U(N)$. In the bulk type IIA string theory dual, this amounts to adding N_f D6-branes wrapping $\mathrm{AdS}_4 \times \mathbb{RP}^3$, which preserve $\mathcal{N}=3$ supersymmetry. The vector model is then obtained by taking $M=0$. The string theory dual would be the "minimal radius" $\mathrm{AdS}_4 \times \mathbb{CP}^3$, supported by the N_f D6-branes and flat Kalb-Ramond B-field with

$$\frac{1}{2\pi\alpha'} \int_{\mathbb{CP}^1} B_{NS} = \frac{N}{k} + \frac{1}{2}. \tag{5.179}$$

In this case, our proposed dual $n=4$ Vasiliev theory in AdS_4 with $\mathcal{N}=3$ boundary condition carries $U(N_f)$ Chan-Paton factors, as does the open string field theory on the D6-

branes. This lead to the natural conjecture that the open-closed string field theory of the D6-branes in the "minimal" $\mathrm{AdS}_4 \times \mathbb{CP}^3$ with flat B-field is the *same* as the $n = 4$ Vasiliev theory, at the level of classical equations of motion. It would be fascinating to demonstrate this directly from type IIA string field theory in $\mathrm{AdS}_4 \times \mathbb{CP}^3$, say using the pure spinor formalism [86, 87, 88].

5.7 Conclusion

In this paper, we proposed the higher spin gauge theories in AdS_4 described by supersymmetric extensions of Vasiliev's system and appropriate boundary conditions that are dual to a large class of supersymmetric Chern-Simons vector models. The parity violating phase θ_0 in Vasiliev theory plays the key role in identifying the boundary conditions that preserve or break certain supersymmetries. In particular, our findings are consistent with the following conjecture: starting with the duality between parity invariant Vasiliev theory and the dual free supersymmetric $U(N)$ vector model at large N, turning on Chern-Simons coupling for the $U(N)$ corresponds to turning on the parity violating phase θ_0 in the bulk, and at the same time induces a change of fermion boundary condition as described in Section 5.5.4. We conjectured that the relation $\theta_0 = \frac{\pi}{2}\lambda$, where $\lambda = N/k$ is the 't Hooft coupling of the boundary Chern-Simons theory, suggested by two-loop perturbative calculation in the field theory and Giveon-Kutasov duality and ABJ self duality, is exact.

Turning on various scalar potential and scalar-fermion coupling in the Chern-Simons vector model amounts to double trace and triple trace deformations, which are dual to deformation of boundary conditions on spin 0 and spin 1/2 fields in the bulk theory. Gauging a flavor symmetry of the boundary theory with Chern-Simons amounts to changing the boundary condition on the bulk spin-1 gauge field from the magnetic boundary condition to a electric-magnetic mixed boundary condition. Consideration of supersymmetry breaking by boundary conditions allowed us to identify precise relations between θ_0, the Chern-Simons level k, and two-point function coefficient $\widetilde{N}$ in $\mathcal{N}=3$ Chern-Simons vector models.

While substantial evidence for the dualities proposed in this paper is provided by the analysis of linear boundary conditions, we have not analyzed in detail the non-linear corrections to the boundary conditions, which are responsible for the triple trace terms needed to preserve supersymmetry. Furthermore, we have not nailed down the bulk theory completely, due to the possible non-constant terms in the function $\theta(X)=\theta_0+\theta_2 X^2+\theta_4 X^4+\cdots$ that controls bulk interactions and breaks parity. It seems that θ_2,θ_4 etc. cannot be removed merely by field redefinition, and presumably contribute to five and higher point functions at bulk tree level, and yet their presence would not affect the preservation of supersymmetry. This non-uniqueness at higher order in the bulk equation of motion is puzzling, as we know of no counterpart of it in the dual boundary CFT. Perhaps clues to resolving this puzzle can be found by explicit computation of say the contribution of θ_2 to the five-point function. It is possible that a thorough analysis of the near boundary behavior of solutions to Vasiliev's equations (via a Graham Fefferman type analysis) could be useful in this regard.

We have also encountered another puzzle that applies to Vasiliev duals of all Chern Simons field theories, not necessarily supersymmetric. Our analysis of the bulk Vasiliev description of the breaking of higher spin symmetry correctly reproduced those double trace terms in the divergence of higher spin currents that involve a scalar field on the RHS. However we were unable to reproduce the terms bilinear in two higher spin currents. The reason for this failure was very general; when acting on a state the higher spin symmetry generators never appear to violate the boundary conditions for any field except the scalar. It would be reassuring to resolve this discrepancy.

The triality between ABJ theory, $n = 6$ Vasiliev theory with $U(M)$ Chan-Paton factors, and type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ suggests a concrete way of embedding Vasiliev theory into string theory. In particular, the $U(M)$ Vasiliev theory is controlled by its *bulk* 't Hooft coupling $\lambda_{bulk} = g^2 M \sim M/N$. We see clear indication from the dual field theory that at strong λ_{bulk}, the nonabelian higher spin particles form color neutral bound states, that are single closed string excitations. Vice versa, in the small radius limit and with near critical amount of flat Kalb-Ramond B-field on $\mathbb{CP}^3$, the type IIA strings should break into multi-particle states of higher spin fields. The dual field theory mechanism for the disintegration of the string is very general, and so should apply more generally to the dual string theory description of any field theory with bifundamental matter, when the rank of one of the gauge groups is taken to be much smaller than the other [42].

It has been argued that the vacuum of the ABJ model spontaneously breaks supersymmetry for $k < N - M$ [75]. Requiring the existence of a supersymmetric vacuum, the maximum value of t'Hooft coupling in a theory with $M \neq N$ is $\frac{N}{k_{min}} = \frac{1}{1-\frac{M}{N}}$. As the radius of the dual AdS space in string units is proportional to a positive power of the t'Hooft coupling, it follows that ABJ theories have a weakly curved string description only in the limit $\frac{M}{N} \to 1$. The recasting of ABJ theory as a Vasiliev theory suggests that it would be interesting, purely within field theory, to study ABJ theory in a power expansion in $\frac{M}{N}$ but nonperturbatively in λ. At $\frac{M}{N} = 0$ this would require a generalization of the results of [37] and [51] to the supersymmetric theory. It may then be possible to systematically correct

[42] We thank K. Narayan for discussions on this point.

this solution in a power series in $\frac{M}{N}$. This would be fascinating to explore.

Perhaps the most surprising recipe in this web of dualities is that the full classical equation of motion of the bulk higher spin gauge theory can be written down explicitly and exactly, thanks to Vasiliev's construction. One of the outstanding questions is how to derive Vasiliev's system directly from type IIA string field theory in $\mathrm{AdS}_4 \times \mathbb{CP}^3$, or to learn about the structure of the string field equations (in AdS) from Vasiliev's equations. As already mentioned, a promising approach is to consider the open-closed string field theory on D6-branes wrapped on $\mathrm{AdS}_4 \times \mathbb{RP}^3$, which should directly reduce to $n = 4$ Vasiliev theory in the minimal radius limit. It would also be interesting to investigate whether - and in what guise - the huge bulk gauge symmetry of Vasiliev's description survives in the bulk string sigma model description of the same system. We leave these questions to future investigation.

5.A Details and explanations related to Section 5.2

5.A.1 Star product conventions and identities

It follows from the definition of the star product that

$$
\begin{aligned}
&y^\alpha * y^\beta = y^\alpha y^\beta + \epsilon^{\alpha\beta}; \quad [y^\alpha, y^\beta]_* = 2\epsilon^{\alpha\beta} \\
&z^\alpha * z^\beta = z^\alpha z^\beta - \epsilon^{\alpha\beta}; \quad [z^\alpha, z^\beta]_* = -2\epsilon^{\alpha\beta} \\
&y^\alpha * z^\beta = y^\alpha z^\beta - \epsilon^{\alpha\beta}; \quad z^\alpha * y^\beta = z^\alpha y^\beta + \epsilon^{\alpha\beta}; \quad [y^\alpha, z^\beta]_* = 0
\end{aligned}
\tag{5.180}
$$

Identical equations (with obvious modifications) apply to the bar variables. Spinor indices are lowered using the ϵ tensor as follows

$$
z_\alpha = z^\beta \epsilon_{\beta\alpha}, \quad \epsilon_{12} = -\epsilon_{21} = \epsilon^{12} = -\epsilon^{21} = 1, \quad \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = -\delta_\alpha^\beta \tag{5.181}
$$

Note that for an arbitrary function f we have

$$
\begin{aligned}
z^\alpha * f &= z^\alpha f + \epsilon^{\alpha\beta}(\partial_{y^\beta} f - \partial_{z^\beta} f) \\
f * z^\alpha &= z^\alpha f + \epsilon^{\alpha\beta}(\partial_{y^\beta} f + \partial_{z^\beta} f)
\end{aligned}
\tag{5.182}
$$

Using (5.182) we the following (anti)commutator

$$
\begin{aligned}
[z^\alpha, f]_* &= -2\epsilon_{\alpha\beta}\partial_{z^\beta} f \\
\{z^\alpha, f\}_* &= 2z^\alpha f + 2\epsilon^{\alpha\beta}\partial_{y^\beta} f
\end{aligned}
\tag{5.183}
$$

It follows from (5.180) that

$$
[z_\alpha, f]_* = -2\frac{\partial f}{\partial z^\alpha}, \quad [y^\alpha, f]_* = 2\epsilon^{\alpha\beta}\frac{\partial f}{\partial y^\beta}, \quad [y_\alpha, f]_* = 2\frac{\partial f}{\partial y^\alpha} \tag{5.184}
$$

Similar expression(with obvious modifications) are true for (anti)commutators with $\bar{y}$ and $\bar{z}$. Substituting $f = K$ into (5.182) and using $\partial_{y^\alpha} K = -z_\alpha K$, one obtains

$$
\{z^\alpha, K\}_* = 0, \quad \text{i.e.} \quad K * z^\alpha * K = -z^\alpha \tag{5.185}
$$

In a similar manner we find

$$\{y^\alpha, K\}_* = 0, \quad \text{i.e.} \quad K * y^\alpha * K = -y^\alpha$$

On the other hand K clearly commutes with $\bar{y}_{\dot{\alpha}}$ and $\bar{z}_{\dot{\alpha}}$. The second line of (5.3) follows immediately from these observations.

The first line of (5.3) is also easily verified.

5.A.2 Formulas relating to ι operation

We present a proof of (5.16)

$$\begin{aligned}
\iota(f * g) &= \Bigg(f(Y,Z) \exp\Bigg[\epsilon^{\alpha\beta} \left(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha} \right) \left(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta} \right) \\
&\qquad\qquad + \epsilon^{\dot{\alpha}\dot{\beta}} \left(\overleftarrow{\partial}_{\bar{y}^{\dot{\alpha}}} + \overleftarrow{\partial}_{\bar{z}^{\dot{\alpha}}} \right) \left(\overrightarrow{\partial}_{\bar{y}^{\dot{\beta}}} - \overrightarrow{\partial}_{\bar{z}^{\dot{\beta}}} \right) \Bigg] g(Y,Z) \Bigg)_{(Y,Z)\to(\tilde{Y},\tilde{Z})} \\
&= f(\tilde{Y},\tilde{Z}) \exp\Bigg[-\epsilon^{\alpha\beta} \left(\overleftarrow{\partial}_{y^\alpha} - \overleftarrow{\partial}_{z^\alpha} \right) \left(\overrightarrow{\partial}_{y^\beta} + \overrightarrow{\partial}_{z^\beta} \right) \\
&\qquad\qquad - \epsilon^{\dot{\alpha}\dot{\beta}} \left(\overleftarrow{\partial}_{y^{\dot{\alpha}}} - \overleftarrow{\partial}_{z^{\dot{\alpha}}} \right) \left(\overrightarrow{\partial}_{y^{\dot{\beta}}} + \overrightarrow{\partial}_{z^{\dot{\beta}}} \right) \Bigg] g(\tilde{Y},\tilde{Z}) \\
&= \iota(g) * \iota(f)
\end{aligned} \tag{5.186}$$

where $(Y,Z) = (y, \bar{y}, z, \bar{z})$ and $(\tilde{Y},\tilde{Z}) = (iy, i\bar{y}, -iz, -i\bar{z}, -idz, -id\bar{z})$.

We now demonstrate that

$$\iota(C * D) = -\iota(D) * \iota(C)$$

if C and D are each one-forms.

$$\begin{aligned}
\iota(C * D) = \iota\left(C_M * D_N dX^M dX^N) \right) &= \iota(D_N) * \iota(C_M)\iota(dX^M)\iota(dX^N) \\
&= -\iota(D_N) * \iota(C_M)\iota(dX^N)\iota(dX^M) = -\iota(D) * \iota(C)
\end{aligned} \tag{5.187}$$

5.A.3 Different Projections on Vasiliev's Master Field

One natural projection one might impose on the Vasiliev master field is to restrict to real fields where reality is defined by

$$\mathcal{A} = \mathcal{A}^* \tag{5.188}$$

This projection preserves the reality of the field strength (i.e. $\mathcal{F}$ is real if $\mathcal{A}$ is). As we will see below, however, the projection (5.188) does not have a natural extension to the supersymmetric Vasiliev theory, and is not the one we will adopt in this paper.

The second 'natural' projection on Vasiliev's master fields is given by

$$\iota(W) = -W, \quad \iota(S) = -S, \quad \iota(B) = K * B * K. \tag{5.189}$$

Note that the various components of $\mathcal{F}$ transform homogeneously under this projection

$$\begin{aligned}
&\iota\left(d_x W + W * W\right) = -\left(d_x W + W * W\right), \\
&\iota\left(d_x \hat{S} + \{W, \hat{S}\}_*\right) = -\left(d_x \hat{S} + \{W, \hat{S}\}_*\right), \\
&\iota\left(\hat{S} * \hat{S}\right) = -\left(\hat{S} * \hat{S}\right),
\end{aligned} \tag{5.190}$$

(the signs in (5.189) were chosen to ensure that all the quantities in (5.190) transform homogeneously). Note also that

$$\iota(B * K) = B * K, \quad \iota(B * \bar{K}) = B * \bar{K}. \tag{5.191}$$

(we have used $K * K = 1$).

As we have explained in the main text, in this paper we impose the projection (5.17) on all fields. (5.17) may be thought of as the product of the projections (5.188) and (5.189). As we have mentioned in the main text $\mathcal{F}$ transforms homogeneously under this truncation (see (5.18)); in components

$$\begin{aligned}
&\iota\left(d_x W + W * W\right)^* = -\left(d_x W + W * W\right),\\
&\iota\left(d_x\hat{S} + \{W, \hat{S}\}_*\right)^* = -\left(d_x\hat{S} + \{W, \hat{S}\}_*\right),\\
&\iota\left(\hat{S} * \hat{S}\right)^* = -\left(\hat{S} * \hat{S}\right).
\end{aligned} \tag{5.192}$$

5.A.4 More about Vasiliev's equations

Expanded in components the first equation in (5.20) reads

$$\begin{aligned}
&d_x W + W * W = 0,\\
&d_x\hat{S} + \{W, \hat{S}\}_* = 0,\\
&\hat{S} * \hat{S} = f_*(B * K)dz^2 + \bar{f}_*(B * \bar{K})d\bar{z}^2.
\end{aligned} \tag{5.193}$$

The second equation reads

$$\begin{aligned}
&d_x B + W * B - B * \pi(W) = 0,\\
&\hat{S} * B - B * \pi(\hat{S}) = 0.
\end{aligned} \tag{5.194}$$

We will now demonstrate that the second equation in (5.20) follows from the first (i.e. that (5.194) follows from (5.193)). Using (5.21) and the first of (5.20) we conclude that

$$d_x\left(f_*(B * K)dz^2 + \bar{f}_*(B * \bar{K})d\bar{z}^2\right) + \hat{A} * \left(f_*(B * K)dz^2 + \bar{f}_*(B * \bar{K})d\bar{z}^2\right) = 0. \tag{5.195}$$

The components of (5.195) proportional to $dxdz^2$ yield,

$$d_x B * K + [W, B * K]_* = 0 \tag{5.196}$$

Multiplying this equation by K from the right and using $K * W * K = \pi(W)$ we find the first of (5.194).

The components of (5.195) proportional to $dxd\bar{z}^2$ yield

$$d_x B * \bar{K} + [W, B * \bar{K}]_* = 0 \tag{5.197}$$

Multiplying this equation by $\bar{K}$ from the right and using $\bar{K} * W * \bar{K} = \bar{K} * W * \bar{K} = \pi(W) =$ (the second step uses the truncation condition (5.11) on W) we once again find the first of (5.194).

The term in (5.195) proportional to $dz^2 d\bar{z}$ and $dzd\bar{z}^2$ may be processed as follows. Let

$$\hat{S} = \hat{S}_z + \hat{S}_{\bar{z}} \tag{5.198}$$

where $\hat{S}_z$ is proportional to dz and $\hat{S}_{\bar{z}}$ is proportional to $d\bar{z}$. The part of (5.195) proportional to $dz^2 d\bar{z}$ yields

$$[S_{\bar{z}}, B * K]_* = 0 \tag{5.199}$$

Multiplying this equation with K from the right and using $K * \hat{S}_{\bar{z}} * K = \pi(\hat{S}_{\bar{z}})$ we find

$$\hat{S}_{\bar{z}} * B - B * \pi(\hat{S}_{\bar{z}}) = 0 \tag{5.200}$$

Finally, the part of (5.195) proportional to $dzd\bar{z}^2$ yields

$$[S_z, B * \bar{K}]_* = 0 \tag{5.201}$$

Multiplying this equation with $\bar{K}$ from the right and using

$$\bar{K} * \hat{S}_z * \bar{K} = \bar{\pi}(\hat{S}_z) = \pi(\hat{S}_z)$$

(where we have used (5.12)) we find

$$\hat{S}_z * B - B * \pi(\hat{S}_z) = 0 \tag{5.202}$$

Adding together (5.200) and (5.202) we find the second of (5.194)

The fact that z and $\bar{z}$ each have only two components, mean that there are no terms in (5.195) proportional to dz^3 or $d\bar{z}^3$, so we have fully analyzed the content of (5.195).

5.A.5 Onshell (Anti) Commutation of components of Vasiliev's Master Field

In this subsection we list some useful commutation and anticommutation relations between the adjoint fields S_z, $S_{\bar{z}}$, $B*K$ and $B*\bar{K}$. The relations we list can be derived almost immediately from Vasiliev's equations; we list them for ready reference

$$\begin{aligned}
&[B*K, B*\bar{K}]_* = 0 \\
&\{S_z, S_{\bar{z}}\}_* = 0 \\
&[S_{\bar{z}}, B*K]_* = 0 \\
&[S_z, B*\bar{K}]_* = 0 \\
&\{S_{\bar{z}}, B*K\}_* = 0 \\
&\{S_z, B*\bar{K}\}_* = 0
\end{aligned} \tag{5.203}$$

The derivation of these equations is straightforward. The first equation follows upon expanding the commutator and noting that $K*B*\bar{K} = \bar{K}*B*K$ (this follows from (5.11) together with the obvious fact that K and $\bar{K}$ commute). The second equation in (5.203) follows upon inserting the decomposition (5.198) into the third equation in (5.193). The third and fourth equations in (5.203)are simply (5.199) and (5.201) rewritten.

The fifth equation in (5.203) may be derived from the third equation as follows

$$\begin{aligned}
&S_{\bar{z}}*B*K = B*K*S_{\bar{z}} \\
&\Rightarrow S_{\bar{z}}*B = B*K*S_{\bar{z}}*K \\
&\Rightarrow S_{\bar{z}}*B = -B*\bar{K}*S_{\bar{z}}*\bar{K} \\
&\Rightarrow S_{\bar{z}}*B*\bar{K} = -B*\bar{K}*S_{\bar{z}}
\end{aligned} \tag{5.204}$$

In the third line of (5.204) we have used the truncation condition (5.11)

The sixth equation in (5.203) is derived in a manner very similar to the fifth equation.

5.A.6 Canonical form of $f(X)$ in Vasiliev's equations

In this subsection we demonstrate that we can use the change of variables $X \to g(X)$ for some odd real function $g(X)$ together with multiplication by an invertible holomorphic even function to put any function $f(X)$ in the form (5.30), at least provided that the function $f(X)$ admits a power series expansion about $X = 0$ and that $f(0) \neq 0$.

An arbitrary function $f(X)$ may be decomposed into its even and odd parts

$$f(X) = f_e(X) + f_o(X)$$

If $f_e(X)$ in invertible then the freedom of multiplication with an even complex function may be used to put $f(X)$ in the form

$$f(X) = 1 + \tilde{f}_o(X)$$

where $\tilde{f}_o(X) = \frac{f_o(X)}{f_e(X)}$. Clearly $\tilde{f}_o(X)$ is an odd function that admits a power series expansion. At least in the sense of a formal power series expansion of all functions, it is easy to convince oneself that any such function may be written in the form $g(X)e^{i\theta(X)}$ where $g(X)$ is an a real odd function and $\theta(X)$ is a real even function. We may now use the freedom of variable redefinitions to work with the variable $g(X)$ instead of X. This redefinition preserves the even nature of $\theta(X)$ and casts $f(X)$ in the form (5.30).

5.A.7 Conventions for $SO(4)$ spinors

Euclidean $SO(4)$ Γ matrices may be chosen as

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \bar{\sigma}_a & 0 \end{pmatrix} \tag{5.205}$$

where $a = 1 \dots 4$ and

$$\sigma_a = (\sigma_i, iI), \quad \bar{\sigma}_a = -\sigma_2 \sigma_a^T \sigma_2 = (\sigma_i, -iI) \tag{5.206}$$

(where $i = 1 \dots 3$ and σ^i are the usual Pauli matrices). In the text below we will often refer to the fourth component of σ^μ as σ^z; in other words

$$\sigma^z = iI$$

(we adopt this cumbersome notation to provide easy passage to different conventions). The chirality matrix $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$ is given by

$$\Gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{5.207}$$

Γ matrices act on the spinors

$$\begin{pmatrix} \chi_\alpha \\ \bar{\zeta}^{\dot{\beta}} \end{pmatrix}$$

whereas the row spinors that multiply Γ from the left have the index structure

$$\begin{pmatrix} \chi^\alpha & \bar{\zeta}_{\dot{\beta}} \end{pmatrix}$$

As a consequence we assign the index structure $(\sigma_a)_{\alpha\dot{\beta}}$ and $\bar{\sigma}^{\dot{\alpha}\beta}$. It is easy to check that

$$[\Gamma_a, \Gamma_b] = 2 \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix} \tag{5.208}$$

where

$$\begin{aligned}&\sigma_{ab} = \frac{1}{2}(\sigma_a\bar{\sigma}_b - \sigma_b\bar{\sigma}_a), \quad \bar{\sigma}_{ab} = \frac{1}{2}(\bar{\sigma}_a\sigma_b - \bar{\sigma}_b\sigma_a) \\ \Rightarrow \quad &\sigma_{ij} = i\epsilon_{ijk}\sigma^k, \quad \bar{\sigma}_{ij} = i\epsilon_{ijk}\bar{\sigma}^k, \quad \sigma_{i4} = -i\sigma_i, \quad \bar{\sigma}_{i4} = i\sigma_i\end{aligned} \tag{5.209}$$

Clearly the index structure above is $(\sigma_{ab})_\alpha{}^\beta$ and $(\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}$. Spinor indices are raised and lowered according to the conventions

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \psi^\alpha = \psi_\beta\epsilon^{\beta\alpha}, \quad \epsilon^{12} = \epsilon_{12} = 1$$

The product of a chiral spinor y^α and an antichiral spinor $\bar{y}^{\dot{\beta}}$ is a vector. By convention we define the associated vector as

$$V_\mu = y^\alpha(\sigma_\mu)_{\alpha\dot{\beta}}\bar{y}^{\dot{\beta}} \tag{5.210}$$

The product of a chiral spinor y with itself is a self dual antisymmetric 2 tensor which we take to be

$$V_{ab} = y^\alpha(\sigma_{ab})_\alpha{}^\beta y_\beta \tag{5.211}$$

Similarly the product of an antichiral spinor with itself is an antiselfdual 2 tensor which we take to be

$$V_{ab} = \bar{y}_{\dot{\alpha}}(\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{y}^{\dot{\beta}} \tag{5.212}$$

5.A.8 AdS$_4$ solution

In this appendix we will show that

$$W_0 = (e_0)_{\alpha\dot{\beta}}y^\alpha\bar{y}^{\dot{\beta}} + (\omega_0)_{\alpha\beta}y^\alpha y^\beta + (\omega_0)_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}} \tag{5.213}$$

with the AdS$_4$ values for the vielbein and spin connection, satisfies the Vasiliev equation

$$d_xW_0 + W_0 * W_0 = 0. \tag{5.214}$$

Substituting (5.213) in (5.214) and collecting terms quadratic in y and $\bar{y}$ we get

$$\begin{aligned}
y^{\alpha}\bar{y}^{\dot{\alpha}}: &\quad d_x e_{\alpha\dot{\beta}} + 4\omega_{\alpha}^{\ \beta} \wedge e_{\beta\dot{\beta}} - 4e_{\alpha\dot{\gamma}} \wedge \omega^{\dot{\gamma}}_{\ \dot{\beta}} = 0 \\
y^{\alpha}y^{\beta}: &\quad d_x\omega_{\alpha}^{\ \beta} - 4\omega_{\alpha}^{\ \gamma} \wedge w_{\gamma}^{\ \beta} - e_{\alpha\dot{\alpha}} \wedge e_{\beta\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} = 0 \\
y^{\dot{\alpha}}y^{\dot{\beta}}: &\quad d_x\omega^{\dot{\alpha}}_{\ \dot{\beta}} + 4\omega^{\dot{\alpha}}_{\ \dot{\gamma}} \wedge \omega^{\dot{\gamma}}_{\ \dot{\beta}} - e_{\alpha\dot{\alpha}} \wedge e_{\beta\dot{\beta}}\epsilon^{\alpha\beta} = 0
\end{aligned} \tag{5.215}$$

Let us consider the Vasiliev gauge transformations generated by

$$\epsilon(x|Y) = C_{1ab}\ (y\sigma_{ab}y) + C_{2ab}\ (\bar{y}\bar{\sigma}_{ab}\bar{y})$$

Under these the vielbein and spin connection changes by

$$\begin{aligned}
\delta e_{\alpha\dot{\alpha}} &= -4C_{1ab}(\sigma_{ab})_{\alpha}^{\ \beta} e_{\beta\dot{\alpha}} - 4C_{2ab}\ e_{\alpha\dot{\beta}}(\bar{\sigma}_{ab})^{\dot{\beta}}_{\ \dot{\alpha}} \\
\delta\omega_{\alpha}^{\ \beta} &= d_x C_{1ab}(\sigma_{ab})_{\alpha}^{\ \beta} - 8C_{1ab}\ \omega_{\alpha}^{\ \gamma}(\sigma_{ab})_{\gamma}^{\ \beta} \\
\delta\omega^{\dot{\alpha}}_{\ \dot{\beta}} &= d_x C_{2ab}(\bar{\sigma}_{ab})^{\dot{\alpha}}_{\ \dot{\beta}} + 8C_{2ab}\ \omega^{\dot{\alpha}}_{\ \dot{\gamma}}(\bar{\sigma}_{ab})^{\dot{\gamma}}_{\ \dot{\beta}}
\end{aligned} \tag{5.216}$$

Notice that these are just the rotation of the vielbeins in the tangent space. The two homogeneous terms in δe are just the rotation by under $SU(2)_L$ and $SU(2)_R$ of $SO(4)$ that acts on the tangent space. As expected under such rotation the spin connection transforms inhomogeneously. Substituting (5.216) in (5.215) it is easily verified that (5.215) transforms homogeneously.

In fact the first equation in (5.215) is just the torsion free condition while the second and third equation expresses the selfdual and anti-selfdual part of curvature two form in term of vielbeins. Substituting the AdS$_4$ values of vielbeins and spin connection (5.36) one can easily check that (5.215) are satisfied.

Converting (5.215) from bispinor notation to $SO(4)$ vector notation using the following conversion

$$e_{\alpha\dot{\beta}} = 2e_a(\sigma_a)_{\alpha\dot{\beta}}\ ,\quad \omega_{\alpha}^{\ \beta} = \frac{1}{16}\omega_{ab}(\sigma_{ab})_{\alpha}^{\ \beta},\quad \omega^{\dot{\alpha}}_{\ \dot{\beta}} = -\frac{1}{16}\omega_{ab}(\sigma_{ab})^{\dot{\alpha}}_{\ \dot{\beta}}, \tag{5.217}$$

we get

$$
\begin{aligned}
T_a &\equiv d_x e_a + \omega_{ab} \wedge e_b = 0 \\
R_{ab} &\equiv d_x \omega_{ab} + \omega_{ac} \wedge \omega_{cb} + 64 e_a \wedge e_b = 0.
\end{aligned}
\tag{5.218}
$$

5.A.9 Exploration of various boundary conditions for scalars in the non abelian theory

The same theory in AdS$_4$ with $\Delta = 2$ boundary condition on the $U(M)$-*singlet* bulk scalar is dual to the critical point of the $SU(N)$ vector model with M flavors and the double trace deformation by $(\bar{\phi}^{ia}\phi_{ia})^2$. Alternatively, this critical point may be defined by introducing a Lagrangian multiplier α and adding the term

$$
\alpha \bar{\phi}^{ia}\phi_{ia} \tag{5.219}
$$

to the Lagrangian of the vector model.[43] As in the case of the $M = 1$ critical vector model, higher spin symmetry is broken by $1/N$ effects. Note that the $SU(M)$ part of the spin-2 current is also broken by $1/N$ effects, i.e. there are no interacting colored massless gravitons, as expected. To see this explicitly from the boundary CFT, let us consider the spin-2 current

$$
(J^{(2)}_{\mu\nu})^a{}_b = \frac{1}{2}\bar{\phi}^{ia}\overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}_\nu \phi_{ib} - 2\partial_{(\mu}\bar{\phi}^{ia}\partial_{\nu)}\phi_{ib} + \delta_{\mu\nu}\partial^\rho\bar{\phi}^{ia}\partial_\rho\phi_{ib}. \tag{5.220}
$$

Using the classical equation of motion

$$
\Box\phi_i = \alpha\phi_i, \tag{5.221}
$$

we have

$$
\partial^\mu (J^{(2)}_{\mu\nu})^a{}_b = (\partial_\nu\alpha)\bar{\phi}^{ia}\phi_{ib} - \alpha\partial_\nu(\bar{\phi}^{ia}\phi_{ib}). \tag{5.222}
$$

[43] The critical point can be conveniently defined using dimensional regularization.

While the $SU(M)$-singlet part of $J_{\mu\nu}$, being the stress-energy tensor, is conserved ($\bar{\phi}^{ia}\phi_{ia}$ is set to zero by α-equation of motion), the $SU(M)$ non-singlet part of $J_{\mu\nu}$ is not conserved, and acquires an anomalous dimension of order $1/N$ at the leading nontrivial order in the $1/N$ expansion. In the bulk, the colored gravitons become massive, and their longitudinal components are supplied by the bound state of the singlet scalar and a colored spin-1 field.

One could also consider the theory in AdS_4 with $\Delta = 2$ boundary condition on *all* bulk scalars, that is, on both the singlet and adjoint of the $SU(M)$ bulk gauge group. The dual CFT is the critical point defined by turning on the double trace deformation $\bar{\phi}^{ia}\phi_{ib}\bar{\phi}^{jb}\phi_{ja}$ and flow to the IR, or by introducing the Lagrangian multiplier $\Lambda_a{}^b$, and the term

$$\Lambda_a{}^b\bar{\phi}^{ia}\phi_{ib} \tag{5.223}$$

in the CFT Lagrangian. Now the classical equations of motion

$$\Box\phi_{ia} = \Lambda_a{}^b\phi_{ib}, \qquad \bar{\phi}^{ia}\phi_{ib} = 0, \tag{5.224}$$

imply the divergence of the colored spin-2 currents

$$\partial^\mu (J^{(2)}_{\mu\nu})^a{}_b = \Lambda_b{}^c\bar{\phi}^{ia}\overleftrightarrow{\partial}_\nu\phi_{ic} - \Lambda_c{}^a\bar{\phi}^{ic}\overleftrightarrow{\partial}_\nu\phi_{ib} = \Lambda_b{}^c(J^{(1)}_\nu)^a{}_c - \Lambda_c{}^a(J^{(1)}_\nu)^c{}_b. \tag{5.225}$$

Once again, the $SU(M)$ non-singlet spin-2 current is no longer conserved. In this case, the colored gravitons in the bulk are massive because their longitudinal component are supplied by the two-particle state of colored scalar and spin-1 fields.

5.B Supersymmetry transformations on bulk fields of spin 0, $\frac{1}{2}$, and 1

We begin by rewriting the magnetic boundary condition on the spin-1 bulk fields in the supersymmetric Vasiliev theory. With the magnetic boundary condition, the 2^{n-1} vector gauge fields are dual to ungauged $U(2^{\frac{n}{2}-1}) \times U(2^{\frac{n}{2}-1})$ "R-symmetry" currents of boundary CFT that rotate the bosonic and fermionic flavors separately. Supersymmetrizing Chern-Simons coupling will generally break this flavor symmetry to a subgroup. We will see this as the violation of magnetic boundary condition by the supersymmetry variation of the bulk spin-1 fields. If we do not gauge the flavor symmetries of the Chern-Simons vector model, then all bulk vector fields should be assigned the magnetic boundary condition. We will see later that in this case only up to $\mathcal{N} = 3$ supersymmetry can be preserved, whereas by relaxing the magnetic boundary condition on some of the bulk vector fields, it will be possible to preserve $\mathcal{N} = 4$ or 6 supersymmetry.

In terms of Vasiliev's master field B which contains the field strength, the general electric-magnetic boundary condition may be expressed as

$$B\Big|_{\mathcal{O}(y^2,\bar{y}^2)} \to z^2 \left[e^{i\beta}(yFy) + e^{-i\beta}(\bar{y}\overline{F}\bar{y})\Gamma\right], \qquad z \to 0, \tag{5.226}$$

where $F \equiv F_{\mu\nu}\sigma^{\mu\nu}$ and its complex conjugate $\overline{F}$ are functions of ψ_i, and are constrained by the linear relation

$$F = -\sigma^z \overline{F} \sigma^z. \tag{5.227}$$

With this choice of boundary condition, the boundary to bulk propagator for the spin-1 components of the B master field is given by the standard one,

$$\begin{aligned} B^{(1)} &= \frac{z^2}{(\vec{x}^2+z^2)^3} e^{-y\Sigma\bar{y}} \left[e^{i\beta}(\lambda \mathbf{x}\sigma^z y)^2 + e^{-i\beta}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma \right] \\ &\equiv \widetilde{B}^{(1)} \left[e^{i\beta}(\lambda \mathbf{x}\sigma^z y)^2 + e^{-i\beta}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma \right]. \end{aligned} \tag{5.228}$$

It indeed obeys (5.227), with F and $\overline{F}$ given by

$$\begin{aligned} F_\alpha{}^\beta &= -(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z)_\alpha(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z)^\beta, \\ \overline{F}_{\dot\alpha}{}^{\dot\beta} &= -(\lambda\sigma^z\vec{x}\cdot\vec{\sigma}\sigma^z)_{\dot\alpha}(\lambda\sigma^z\vec{x}\cdot\vec{\sigma}\sigma^z)^{\dot\beta} = -(\lambda\vec{x}\cdot\vec{\sigma})_{\dot\alpha}(\lambda\vec{x}\cdot\vec{\sigma})^{\dot\beta}, \end{aligned} \tag{5.229}$$

and

$$(\sigma^z\overline{F}\sigma^z)_\alpha{}^\beta = -(\lambda\vec{x}\cdot\vec{\sigma})_{\dot\alpha}(\lambda\vec{x}\cdot\vec{\sigma})^{\dot\beta}(\sigma^z)_\alpha{}^{\dot\alpha}(\sigma^z)^\beta{}_{\dot\beta} = (\lambda\vec{x}\cdot\vec{\sigma}\sigma^z)_\alpha(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z)^\beta = -F_\alpha{}^\beta. \tag{5.230}$$

In the next four subsections, we give the explicit formulae for the supersymmetry variation δ_ϵ (i.e. spin 3/2 gauge transformation of Vasiliev's system) of bulk fields of spin $0, 1/2, 1$, sourced by boundary currents of spin $0, 1/2, 1$.

5.B.1 δ_ϵ: spin 1 $\to$ spin $\frac{1}{2}$

Let us start with the B master field sourced by a spin-1 boundary current at $\vec{x}=0$, i.e. the spin-1 boundary to bulk propagator $B^{(1)}(x|Y)$, and consider its variation under supersymmetry, which is generated by $\epsilon(x|Y)$ of degree one in $Y=(y,\bar{y})$:

$$\begin{aligned} \delta_\epsilon B^{(1)}(x|Y) = &-\epsilon * e^{i\beta}(\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)} + e^{i\beta}(\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)} * \pi(\epsilon) \\ &- \epsilon * e^{-i\beta}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\widetilde{B}^{(1)} + e^{-i\beta}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\widetilde{B}^{(1)} * \pi(\epsilon). \end{aligned} \tag{5.231}$$

Carrying out the $*$ products explicitly, we find

$$
\begin{aligned}
&-\epsilon*(\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)}+(\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)}*\pi(\epsilon)\\
&=-(\Lambda y+\overline{\Lambda}\bar{y})*(\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)}+(\lambda\mathbf{x}\sigma^z y)^2\widetilde{B}^{(1)}*(-\Lambda y+\overline{\Lambda}\bar{y})\\
&=-\{y_\alpha,(\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}\}_*\{\Lambda^\alpha,\lambda^\beta\lambda^\gamma\}-[y_\alpha,(\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}]_*[\Lambda^\alpha,\lambda^\beta\lambda^\gamma]\\
&\quad-[\bar{y}_{\dot\alpha},(\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}]_*\{\overline{\Lambda}^{\dot\alpha},\lambda^\beta\lambda^\gamma\}-\{\bar{y}_{\dot\alpha},(\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}\}_*[\overline{\Lambda}^{\dot\alpha},\lambda^\beta\lambda^\gamma]\\
&=-2\{\Lambda y,\lambda^\beta\lambda^\gamma\}(\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}-2[\Lambda\partial_y,\lambda^\beta\lambda^\gamma](\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}\\
&\quad-2\{\overline{\Lambda}\partial_{\bar{y}},\lambda^\beta\lambda^\gamma\}(\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}-2[\overline{\Lambda}\bar{y},\lambda^\beta\lambda^\gamma](\mathbf{x}\sigma^z y)_\beta(\mathbf{x}\sigma^z y)_\gamma\widetilde{B}^{(1)}\\
&=2\{\overline{\Lambda}\Sigma y-\Lambda y,(\lambda\mathbf{x}\sigma^z y)^2\}\widetilde{B}^{(1)}+2[\Lambda\Sigma\bar{y}-\overline{\Lambda}\bar{y},(\lambda\mathbf{x}\sigma^z y)^2]\widetilde{B}^{(1)}-4[(\mathbf{x}\sigma^z\Lambda)_\beta,\lambda^\beta(\lambda\mathbf{x}\sigma^z y)]\widetilde{B}^{(1)},
\end{aligned}
\tag{5.232}
$$

and

$$
\begin{aligned}
&-\epsilon*(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\widetilde{B}^{(1)}+(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\widetilde{B}^{(1)}*\pi(\epsilon)\\
&=-2\{\Lambda y,\lambda^\beta\lambda^\gamma\Gamma\}(\sigma^z\mathbf{x}\sigma^z\bar{y})_\beta(\sigma^z\mathbf{x}\sigma^z\bar{y})_\gamma\widetilde{B}^{(1)}-2[\Lambda\partial_y,\lambda^\beta\lambda^\gamma\Gamma](\sigma^z\mathbf{x}\sigma^z\bar{y})_\beta(\sigma^z\mathbf{x}\sigma^z\bar{y})_\gamma\widetilde{B}^{(1)}\\
&\quad-2\{\overline{\Lambda}\partial_{\bar{y}},\lambda^\beta\lambda^\gamma\Gamma\}(\sigma^z\mathbf{x}\sigma^z\bar{y})_\beta(\sigma^z\mathbf{x}\sigma^z\bar{y})_\gamma\widetilde{B}^{(1)}-2[\overline{\Lambda}\bar{y},\lambda^\beta\lambda^\gamma\Gamma](\sigma^z\mathbf{x}\sigma^z\bar{y})_\beta(\sigma^z\mathbf{x}\sigma^z\bar{y})_\gamma\widetilde{B}^{(1)}\\
&=2\{\overline{\Lambda}\Sigma y-\Lambda y,(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma\}\widetilde{B}^{(1)}+2[\Lambda\Sigma\bar{y}-\overline{\Lambda}\bar{y},(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})^2\Gamma]\widetilde{B}^{(1)}-4\{(\sigma^z\mathbf{x}\sigma^z\overline{\Lambda})_\beta,\lambda^\beta(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})\Gamma\}\widetilde{B}^{(1)}.
\end{aligned}
\tag{5.233}
$$

Note that the commutators and anti-commutators in above formula are due to the ψ_i-dependence only, and do not involve $*$ product. $\delta_\epsilon B^{(1)}$ contains supersymmetry variation of fields of spin 1/2 and 3/2. We will focus on the variation spin 1/2 fields, since they can be subject to a family of different boundary conditions, corresponding to turning on fermionic double trace deformations (i.e. (fermion singlet)^2) in the boundary CFT. So we restrict to terms linear in $(y,\bar y)$,

$$\begin{aligned}&\delta B^{(1)}\big|_{\mathcal{O}(y,\bar y)} = -4[(\mathbf{x}\sigma^z\Lambda)_\beta, \lambda^\beta(\lambda\mathbf{x}\sigma^z y)]\widetilde{B}^{(1)} - 4\{(\sigma^z\mathbf{x}\sigma^z\overline{\Lambda})_\beta, \lambda^\beta(\lambda\sigma^z\mathbf{x}\sigma^z\bar y)\Gamma\}\widetilde{B}^{(1)}\\ &\to -4e^{i\beta}\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^3}[(\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)_\beta, \lambda^\beta(\lambda\vec{x}\cdot\vec{\sigma}\sigma^z y)] + 4e^{-i\beta}\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^3}[(\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_+)_\beta, \lambda^\beta(\lambda\vec{x}\cdot\vec{\sigma}\bar y)]\Gamma\end{aligned} \tag{5.234}$$

where in the second line we kept the leading terms, of order $z^{\frac{3}{2}}$, in the $z\to 0$ limit.

5.B.2 δ_ϵ: spin $\frac{1}{2}\to$ spin 1

The general conformally invariant boundary condition on spin 1/2 fermions, in terms of Vasiliev's B master field, takes the form

$$B\big|_{\mathcal{O}(y,\bar y)} \to z^{\frac{3}{2}}\left[e^{i\alpha}(\chi y) - \Gamma e^{-i\alpha}(\bar\chi\bar y)\right], \tag{5.235}$$

Here χ and its complex conjugate $\bar\chi$ are chiral and anti-chiral spinors that are odd functions of the Grassmannian variables ψ_i. They are further constrained by the linear relation

$$\chi = \sigma^z\bar\chi. \tag{5.236}$$

α is generally a linear operator that acts on the vector space spanned by odd monomials in the ψ_i's, i.e. it assigns phase angles to fermions in the bulk R-symmetry multiplet. A choice of the spin-1/2 fermion boundary condition is equivalent to a choice of the "phase angle" operator α.

The fermion boundary to bulk propagator that satisfies the above boundary condition is:

$$\begin{aligned}B^{(\frac{1}{2})} &= \frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}e^{-y\Sigma\bar y}\left[e^{i\alpha}(\lambda\mathbf{x}\sigma^z y) - \Gamma e^{-i\alpha}(\lambda\sigma^z\mathbf{x}\sigma^z\bar y)\right]\\ &\equiv \left[e^{i\alpha}(\lambda\mathbf{x}\sigma^z y) - \Gamma e^{-i\alpha}(\lambda\sigma^z\mathbf{x}\sigma^z\bar y)\right]\widetilde{B}^{(\frac{1}{2})}.\end{aligned} \tag{5.237}$$

Here the linear operator α is understood to act on λ only, the latter being an odd function of ψ_i's.

Next, we make super transformation on the fermion boundary to bulk propagator. The supersymmetry transformation reads

$$\begin{aligned}\delta B^{(\frac{1}{2})} = &- e^{i\alpha}\epsilon * (\lambda \mathbf{x}\sigma^z y)\widetilde{B}^{(\frac{1}{2})} + e^{i\alpha}(\lambda \mathbf{x}\sigma^z y)\widetilde{B}^{(\frac{1}{2})} * \pi(\epsilon) \\ &- e^{-i\alpha}\epsilon * (\lambda\sigma^z \mathbf{x}\sigma^z \bar{y})\Gamma\widetilde{B}^{(\frac{1}{2})} + e^{-i\alpha}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})\Gamma\widetilde{B}^{(\frac{1}{2})} * \pi(\epsilon),\end{aligned} \tag{5.238}$$

where $\epsilon = \Lambda y + \bar{\Lambda}\bar{y}$, Λ is an odd supersymmetry parameter η multiplied by an odd function of the ψ_i's. η in particular anti-commutes with all ψ_i's, and therefore anti-commutes with λ which involves an odd number of ψ_i's.

Carrying out the $*$ algebra, we have

$$\begin{aligned}&-\epsilon * (\lambda\mathbf{x}\sigma^z y)\widetilde{B}^{(\frac{1}{2})} + (\lambda\mathbf{x}\sigma^z y)\widetilde{B}^{(\frac{1}{2})} * \pi(\epsilon) \\ &= 2\{\overline{\Lambda}\Sigma y - \Lambda y, (\lambda\mathbf{x}\sigma^z y)\}\widetilde{B}^{(\frac{1}{2})} + 2[\Lambda\Sigma\bar{y} - \overline{\Lambda}\bar{y}, (\lambda\mathbf{x}\sigma^z y)]\widetilde{B}^{(\frac{1}{2})} - 2[(\mathbf{x}\sigma^z\Lambda)_\beta, \lambda^\beta]\widetilde{B}^{(\frac{1}{2})},\end{aligned} \tag{5.239}$$

and

$$\begin{aligned}&-\epsilon * (\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})\Gamma\widetilde{B}^{(\frac{1}{2})} + (\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})\Gamma\widetilde{B}^{(\frac{1}{2})} * \pi(\epsilon) \\ &= 2\{\overline{\Lambda}\Sigma y - \Lambda y, (\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})\Gamma\}\widetilde{B}^{(\frac{1}{2})} + 2[\Lambda\Sigma\bar{y} - \overline{\Lambda}\bar{y}, (\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})\Gamma]\widetilde{B}^{(\frac{1}{2})} - 2\{(\sigma^z\mathbf{x}\sigma^z\overline{\Lambda})_\beta, \lambda^\beta\Gamma\}\widetilde{B}^{(\frac{1}{2})}.\end{aligned} \tag{5.240}$$

The supersymmetry variation of the spin-1 field strengths are extracted from $\mathcal{O}(y^2, \bar{y}^2)$ terms in $\delta B^{(\frac{1}{2})}$, namely

$$\begin{aligned}&\delta_\epsilon B^{(\frac{1}{2})}(x|Y)\big|_{\mathcal{O}(y^2,\bar{y}^2)} = 2\{\overline{\Lambda}\Sigma y - \Lambda y, e^{i\alpha}(\lambda\mathbf{x}\sigma^z y)\}\widetilde{B}^{(\frac{1}{2})} - 2[\Lambda\Sigma\bar{y} - \overline{\Lambda}\bar{y}, \Gamma e^{-i\alpha}(\lambda\sigma^z\mathbf{x}\sigma^z\bar{y})]\widetilde{B}^{(\frac{1}{2})} \\ &\to -4\frac{z^2}{(\vec{x}^2+z^2)^3}\{\Lambda_0\vec{x}\cdot\vec{\sigma}\sigma^z y, e^{i\alpha}(\lambda\sigma^z\vec{x}\cdot\vec{\sigma}y)\} - 4\frac{z^2}{(\vec{x}^2+z^2)^3}[\Lambda_0\vec{x}\cdot\vec{\sigma}\bar{y}, \Gamma e^{-i\alpha}(\lambda\vec{x}\cdot\vec{\sigma}\bar{y})].\end{aligned} \tag{5.241}$$

In the second line, we have taken the small z limit and kept the leading terms, of order z^2.

5.B.3 δ_ϵ: spin $\frac{1}{2} \to$ spin 0

The supersymmetry variation of the scalar field due to a spin-$\frac{1}{2}$ fermionic boundary source is extracted from $\delta_\epsilon B^{(\frac{1}{2})}$ of the previous subsection, restricted to $y = \bar{y} = 0$:

$$\begin{aligned}
\delta_\epsilon B^{(\frac{1}{2})}\big|_{y,\bar{y}=0}(\vec{x}, z) &= -2[(\mathbf{x}\sigma^z\Lambda)_\beta, e^{i\alpha}\lambda^\beta]\widetilde{B}^{(\frac{1}{2})} - 2\Gamma[(\sigma^z\mathbf{x}\sigma^z\overline{\Lambda})_\beta, e^{-i\alpha}\lambda^\beta]\widetilde{B}^{(\frac{1}{2})} \\
&\quad + 2z^{-\frac{1}{2}}\Gamma[(\sigma^z\mathbf{x}\Lambda_+)_\beta, e^{-i\alpha}\lambda^\beta]\widetilde{B}^{(\frac{1}{2})} - 2z^{\frac{1}{2}}\Gamma[(\sigma^z\mathbf{x}\Lambda_-)_\beta, e^{-i\alpha}\lambda^\beta]\widetilde{B}^{(\frac{1}{2})} \\
&= 2(e^{i\alpha} + \Gamma e^{-i\alpha})\frac{z}{(\vec{x}^2 + z^2)^2}[(\sigma^z\vec{x}\cdot\vec{\sigma}\Lambda_+)_\beta, \lambda^\beta] - 2(e^{i\alpha} - \Gamma e^{-i\alpha})\frac{z^2}{(\vec{x}^2 + z^2)^2}[(\Lambda_+)_\beta, \lambda^\beta] \\
&\quad - 2(e^{i\alpha} - \Gamma e^{-i\alpha})\frac{z^2}{(\vec{x}^2 + z^2)^2}[(\vec{x}\cdot\vec{\sigma}\sigma^z\Lambda_-)_\beta, \lambda^\beta] + \mathcal{O}(z^3).
\end{aligned} \tag{5.242}$$

In the last two lines, α as a linear operator is understood to act on λ only (and not on $\Lambda_\pm$).

5.B.4 δ_ϵ: spin $0 \to$ spin $\frac{1}{2}$

The general conformally invariant linear boundary condition on the bulk scalars $B^{(0)}(\vec{x}, z) = B(\vec{x}, z|y = \bar{y} = 0)$ may be expressed as

$$B^{(0)}(\vec{x}, z) = (e^{i\gamma} + \Gamma e^{-i\gamma})\tilde{f}_1 z + (e^{i\gamma} - \Gamma e^{-i\gamma})\tilde{f}_2 z^2 + \mathcal{O}(z^3) \tag{5.243}$$

in the limit $z \to 0$. Here $\tilde{f}_1, \tilde{f}_2$ are real and even function in ψ_i, and are subject to a set of linear relations that eliminate half of their degrees of freedom. The phase γ is generally a linear operator acting on the space spanned by even monomials in the ψ_i's (analogously to α in the fermion boundary condition). We will determine our choice of γ and the linear constraints on $\tilde{f}_{1,2}$ later.

The boundary-to-bulk propagator for the scalar components of the B master field, subject to the above boundary condition, is now written as

$$B^{(0)} = f_1(\psi)\widetilde{B}^{(0)}_{\Delta=1} + f_2(\psi)\widetilde{B}^{(0)}_{\Delta=2}, \tag{5.244}$$

where

$$f_1(\psi) = (e^{i\gamma} + \Gamma e^{-i\gamma})\tilde{f}_1(\psi), \qquad f_2(\psi) = (e^{i\gamma} - \Gamma e^{-i\gamma}))\tilde{f}_2(\psi). \tag{5.245}$$

A straightforward calculation gives the supersymmetry variation of the spin-$\frac{1}{2}$ fermion due to a scalar boundary source at $\vec{x} = 0$,

$$\begin{aligned}
\delta_\epsilon \widetilde{B}^{(0)}(\vec{x}, z)\big|_{\mathcal{O}(y,\bar{y})} &\to -4\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}\{\Lambda_0\sigma^z\vec{x}\cdot\vec{\sigma}y, f_1\} - 4\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}[\Lambda_0\vec{x}\cdot\vec{\sigma}\bar{y}, f_1] \\
&\quad + 2\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}[\Lambda_+\sigma^z\bar{y}, f_2] + 2\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}\{\Lambda_+ y, f_2\} \\
&= -4\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}\left(e^{i\gamma}\{\Lambda_0\sigma^z\vec{x}\cdot\vec{\sigma}y, \tilde{f}_1\} - \Gamma e^{-i\gamma}[\Lambda_0\sigma^z\vec{x}\cdot\vec{\sigma}y, \tilde{f}_1] + e^{i\gamma}[\Lambda_0\vec{x}\cdot\vec{\sigma}\bar{y}, \tilde{f}_1] - \Gamma e^{-i\gamma}\{\Lambda_0\vec{x}\cdot\vec{\sigma}\bar{y}, \tilde{f}_1\}\right) \\
&\quad + 2\frac{z^{\frac{3}{2}}}{(\vec{x}^2+z^2)^2}\left(e^{i\gamma}[\Lambda_+\sigma^z\bar{y}, \tilde{f}_2] + \Gamma e^{-i\gamma}\{\Lambda_+\sigma^z\bar{y}, \tilde{f}_2\} + e^{i\gamma}\{\Lambda_+ y, \tilde{f}_2\} + \Gamma e^{-i\gamma}[\Lambda_+ y, \tilde{f}_2]\right).
\end{aligned} \tag{5.246}$$

We have taken the small z limit, and kept terms of order $z^{\frac{3}{2}}$. Again, in the last two lines γ as a linear operator should be understood as acting on $\tilde{f}_{1,2}(\psi)$ only and not on Λ.

5.C The bulk dual of double trace deformations and Chern Simons Gauging

5.C.1 Alternate and Regular boundary conditions for scalars in AdS$_{d+1}$

In this section we review the AdS/CFT implementation alternate and regular boundary conditions for scalars, in the presence of multitrace deformations. The material reviewed here is well known (see e.g. [64, 76, 77, 78, 79, 65] - we most closely follow the approach of the paper [77]); our brief review focuses on aspects we will have occasion to use in the main text of our paper.

Multi-trace deformations in large N field theories

In this brief subsection we will address the following question: how is the generating function of correlators of a large N field theory modified by the addition of a multi-trace deformation to the action of the theory?

Consider any large N field theory whose single trace operators are denoted by O_i. Let $W(J)$ denote the generating function of correlators[44]

$$\langle e^{J_i O_i} \rangle = e^{-W[J_i]}. \tag{5.248}$$

Note that $W[J_i]$ is of order N^2 in a matrix type large N theory, while it is of order N in a vector type large N theory. For formal purposed below we will find it useful to Legendre transform W to define an effective action for the operators O_i

$$I[O^i] = W[J_i] + O^i J_i. \tag{5.249}$$

$I[O^i]$ is a function only of O^i (and not of J_i) in the following sense. The RHS of (5.249) is viewed as an action for the dynamical variable J_i. The equation of motion for J_i follows from varying this action and is

$$\frac{\partial W}{\partial J_i} = -O^i. \tag{5.250}$$

The RHS of (5.249) is evaluated with the onshell value of J_i.

[44] More precisely this equation should have read

$$\langle e^{\int d^d x J_i(x) O^i(x)} \rangle = e^{-W[J_i(x)]}. \tag{5.247}$$

However for ease of readability, in all the formal discussions of this section we will use compact notation in which we suppress the position dependence of operators and fields, and do not explicitly indicate integration.

$I[O^i]$ plays the role of the effective action for the trace operators O^i. In the large N limit the dynamics of the operators O^i is generated by the classical dynamics of the action $I(O^i)$.

Of course $W[J^i]$ may equally be thought of as the Legendre transform of $I[O^i]$

$$W[J_i] = I[O^i] - O^i J_i, \tag{5.251}$$

where O^i is the function of J^i obtained by solving the equation of motion

$$\frac{\partial I}{\partial O^i} = J_i. \tag{5.252}$$

Now let us suppose that the action S of the original large N field theory is deformed by the addition of a multitrace term $S \to S + P(O^i)$ where $P(O^i)$ is an arbitrary function of O^i. The effective action for this deformed theory is simply given by $\tilde{I}(O^i)$

$$\tilde{I}(O^i) = I(O^i) + P(O^i). \tag{5.253}$$

The generating function of correlators of the deformed theory is once again given by the Legendre transform (5.251) with $I[O^i]$ replaced by $\tilde{I}[O^i]$.

Bulk dual to multi trace deformations in regular and alternate quantization

Consider a real scalar field propagating in AdS$_{d+1}$ according to the action

$$S = \frac{1}{2}\int d^{d+1}x\sqrt{g}\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right). \tag{5.254}$$

It is well known that these scalars admit two distinct conformally invariant boundary conditions - sometimes referred to as alternate and standard quantization - in the mass range $-\left(\frac{d^2}{4} - 1\right) > m^2 > -\frac{d^2}{4}$. In this subsection we will review the very well known rules for the computation of correlation functions for scalars with alternate and standard boundary conditions.

The action (5.254) is ambiguous as it generically receives divergent contributions from the boundary, as we now explain. We use coordinates so that the metric of AdS space is given by (5.33). Near $z = 0$ the general solution to the equation motion from (5.254) takes the form

$$\phi = \frac{\phi_1 z^{\frac{d}{2}-\zeta}}{2\zeta} + \phi_2 z^{\frac{d}{2}+\zeta}, \tag{5.255}$$

where ζ is the positive root of the equation $\zeta^2 = m^2 + \frac{d^2}{4}$. Let us cut of the action (5.254) at a small value, z_c of the coordinate z. Onshell (5.254) evaluates to

$$S = -\frac{1}{2}\int d^d x \frac{1}{z_c^{d-1}}\, \phi\, \partial_z \phi, \tag{5.256}$$

where the integral is evaluated over the boundary surface $z = z^c$. It is easily verified that the action S has a divergence proportional to $z_c^{2\zeta}$ when evaluated on the generic solution (5.255). To cure this divergence we supplement (5.254) with a diffeomorphically invariant boundary action for the d dimensional boundary field $\phi(z_c, x)$

$$\delta S = \frac{1}{2}\int d^d x \sqrt{g}\left(\frac{d}{2} - \zeta\right)\phi^2 \tag{5.257}$$

where, once again, the integral is taken over the boundary surface $z = z_c$ and g is the induced metric on this boundary. It is easily verified that

$$S + \delta S = -\frac{1}{2}\int d^d x \phi_1(x)\phi_2(x). \tag{5.258}$$

Regularity in the interior of AdS relates ϕ_2 to ϕ_1. The relationship is clearly linear and so takes the form

$$\phi_2(x) = \int d^d x G(x-y)\phi_1(y). \tag{5.259}$$

In the rest of this subsection we use abbreviated notation so that (5.258) is written as $S = -\frac{1}{2}\phi_1\phi_2$ and (5.259) is written as $\phi_2 = G\phi_1$. It follows that the onshell action is given by

$$S = -\frac{1}{2}\phi_1 G \phi_1. \tag{5.260}$$

In the case of alternate quantization the boundary action (5.260), thought of as a functional of the dynamical field $\phi_1 = \lim_{z_c \to 0} \frac{\phi}{z_c^{\frac{d}{2}-\zeta}}$, is identified with the single trace effective action $I[O]$ defined in (5.249). The generator of correlators of this theory is obtained by coupling $\phi_1 = \frac{\phi}{z_c^{\frac{d}{2}-\zeta}}$ to a source J:

$$S = -\frac{1}{2}\phi_1 G \phi_1 - J\phi_1. \tag{5.261}$$

The resulting equation of motion for ϕ_1 yields

$$G\phi_1 = -J. \tag{5.262}$$

Integrating out ϕ_1 we find the action

$$S = JG^{-1}J.$$

It follows that the two point function of the dual operator is $-G^{-1}$. It also follows from (5.262) that

$$\phi_2 = -J.$$

in particular ϕ_2 vanishes wherever J vanishes. Consequently, alternate quantization is associated with the boundary condition $\phi_2 = 0$.

The multi trace deformation $P(O)$ of the dual theory is implemented, in alternate quantization, by adding the term $P(\phi_1)$to the boundary effective action (5.260), in perfect imitation of (5.253). Correlation functions of the deformed theory are obtained by the Legendre transform of this augmented boundary action. The resultant equation of motion is

$G\phi_1 + J - P'(\phi_1) = 0$ yields the bulk boundary conditions

$$\phi_2 + J - P'(\phi_1) = 0.$$

In the case of regular quantization we supplement the action (5.260) with an additional degree of freedom $\tilde{\phi}_2$ so that the full boundary action takes the form

$$S = -\frac{1}{2}\phi_1 G\phi_1 + \tilde{\phi}_2\phi_1. \tag{5.263}$$

The dynamical field ϕ_1 is then integrated out using its equation of motion

$$G\phi_1 = \tilde{\phi}_2. \tag{5.264}$$

On shell, therefore $\tilde{\phi}_2 = \phi_2$. The resultant action

$$S = \frac{1}{2}\tilde{\phi}_2 G^{-1}\tilde{\phi}_2 \tag{5.265}$$

as a function of $\tilde{\phi}_2$ is identified with $I(O)$ in (5.249). The generator of correlators of the theory is obtained by coupling $\tilde{\phi}_2$ to a source J

$$S = \frac{1}{2}\tilde{\phi}_2 G^{-1}\tilde{\phi}_2 - J\tilde{\phi}_2\,,$$

and then integrating this field out according to its equations of motion. This allows us, in particular, to identify the two point function of the dual theory with G. Note also that the resultant equation of motion, $G^{-1}\tilde{\phi}_2 = J$ implies

$$\phi_1 = J,$$

so that ϕ_1 vanishes wherever J vanishes. In other words standard quantization is associated with the boundary condition $\phi_1 = 0$. The multitrace deformation $P(O)$ of the dual theory is implemented, in standard quantization, by adding $P(\tilde{\phi}_2)$ to the action (5.265). The resultant boundary condition is

$$\phi_1 - J + P'(\phi_2) = 0.$$

Marginal multitrace deformation with two scalar field in opposite quantization

Consider two scalar fields in AdS$_4$, ϕ and χ, with ϕ quantized with alternate quantization and χ with regular quantization. In the compact notation defined in earlier subsection, the generating function of correlation function of the dual field theory deformed by double trace operator $\tan\theta_0 O_1 O_2$ is

$$S = -\frac{1}{2}G\phi_1^2 - \frac{1}{2}G\chi_1^2 + \chi_1\tilde{\chi}_2 - J_1\phi_1 - J_2\tilde{\chi}_2 + \tan\theta_0\tilde{\chi}_2\phi_1. \tag{5.266}$$

The action is linear in $\tilde{\chi}_2$; the equation of motion for this field immediately yields

$$J_2 = \frac{1}{\cos\theta_0}(\sin\theta_0\phi_1 + \cos\theta_0\chi_1). \tag{5.267}$$

Using (5.267) to eliminate ϕ_1 in favor of χ_1, S simplifies to a function of ϕ_1. The resultant equation of motion yields

$$J_1 = -\frac{1}{\cos\theta_0}G(\cos\theta_0\phi_1 - \sin\theta_0\chi_1). \tag{5.268}$$

Using $G\phi_1 = \phi_2$ and $G\chi_1 = \chi_2$, (5.268) may be rewritten as

$$J_1 = -\frac{1}{\cos\theta_0}(\cos\theta_0\phi_2 - \sin\theta_0\chi_2). \tag{5.269}$$

Upon setting $J_1 = J_2 = 0$, (5.267) and (5.269) express the boundary conditions of the trace deformed model. These boundary conditions may, most succinctly be expressed as follows. Let us define new 'rotated' bulk fields

$$\phi' = \cos\theta_0\phi - \sin\theta_0\chi, \quad \chi' = \sin\theta_0\phi + \cos\theta_0\chi.$$

Note that the rotated fields have same bulk action as the original fields. The boundary conditions (5.267) and (5.269) reduce to

$$\phi_2' = 0, \quad \chi_1' = 0.$$

In summary dual to the double trace deformed field theory has the same action as well as boundary conditions for ϕ' and χ' as the dual to the undeformed theory had for ϕ and χ. Despite this fact, the double trace deformed theory is *not* field redefinition equivalent to the original theory. This can be seen in many ways. Most simply, the full action (5.266) does not have a simple rotational invariance, and does not take a simple form when re-expressed in terms of ϕ' and χ'. This lack of equivalence also shows itself up in the generator of two point functions of the operators dual to ϕ' and χ'. This generating function is obtained by plugging (5.267) and (5.268) into (5.266); we find

$$-S = -\cos^2\theta_0 \frac{J_1^2}{2G} + \cos^2\theta_0 \frac{J_2^2 G}{2} + \sin\theta_0 \cos\theta_0 J_1 J_2. \tag{5.270}$$

The fact that θ_0 does not disappear from (5.270) demonstrates the lack of equivalence of the trace deformed model from the trace undeformed model ($\theta_0 = 0$). Note in particular that the double trace deformed theory has a contact cross two point function

$$\langle O_\phi(x) O_\chi(y)\rangle = \sin\theta_0 \cos\theta_0 \delta(x-y),$$

which is absent in the trace undeformed theory. On the other hand the direct correlators $\langle O_\phi(x) O_\phi(y)\rangle$ and $\langle O_\chi(x) O_\chi(y)\rangle$ have the same spacetime structure in the deformed and undeformed theories, but have different normalizations.

5.C.2 Gauging a $U(1)$ symmetry

Let us begin with a three dimensional CFT with a $U(1)$ global symmetry, generated by the current J_i, where i is the three-dimensional vector index. This theory will be referred to as CFT_∞, as opposed to the theory obtained by gauging the $U(1)$ with Chern-Simons gauge field at level k, which we refer to as CFT_k. Suppose CFT_∞ is dual to a weakly coupled gravity theory in AdS_4. The global $U(1)$ current J_i of the boundary CFT is dual to a gauge field A_μ in the bulk. The two-derivative part of the bulk action for the gauge field is

$$\frac{1}{4}\int \frac{d^3\vec{x}dz}{z^4} F_{\mu\nu}F^{\mu\nu} = \int d^3\vec{x}dz \left(\frac{1}{2}F_{zi}F_{zi} + \frac{1}{4}F_{ij}F_{ij}\right). \tag{5.271}$$

Working in the radial gauge $A_z = 0$, we have

$$F_{zi} = \partial_z A_i, \qquad F_{ij} = \partial_i A_j - \partial_j A_i. \tag{5.272}$$

Consider the linearized, i.e. free, equation of motion

$$(\partial_z^2 + \partial_j^2)A_i - \partial_i\partial_j A_j = 0, \tag{5.273}$$

together with the constraint

$$\partial_z\partial_i A_i = 0. \tag{5.274}$$

Near the boundary, a solution to the equation of motion has two possible asymptotic behaviors, $A_i \sim z + \mathcal{O}(z^2)$, or $A_i \sim 1 + \mathcal{O}(z^2)$. Equivalently, they can be expressed in gauge invariant form as the magnetic boundary condition

$$F_{ij}\big|_{z=0} = 0, \tag{5.275}$$

and the electric boundary condition

$$F_{zi}\big|_{z=0} = 0, \tag{5.276}$$

respectively. With the magnetic boundary condition, A_μ is dual to a $U(1)$ global current in the boundary CFT, i.e. CFT_∞. The family of CFT_k, on the other hand, is dual to the same bulk theory with the mixed boundary condition (still conformally invariant)

$$\left.\left(\frac{1}{2}\epsilon_{ijk}F_{jk} + \frac{i\alpha}{k}F_{zi}\right)\right|_{z=0} = 0. \tag{5.277}$$

Here α is a constant. It will be determined in terms of the two-point function of the current J_i.

Let us now solve the bulk Green's function with the mixed boundary condition. The bulk linearized equation of motion with a point source at $z = z_0$, after a Fourier transformation in the boundary coordinates $\vec{x}$, is

$$(\partial_z^2 - p^2)A_i + p_i p_j A_j = \delta(z - z_0)\xi_i. \tag{5.278}$$

Due to the constraint (5.274), the source ξ_i is restricted by $p_i \xi_i = 0$. The boundary condition is

$$\left(\epsilon_{ijk} p_j A_k + \frac{\alpha}{k}\partial_z A_i\right)\Big|_{z=0} = 0. \tag{5.279}$$

Without loss of generality, let us consider the case $\vec{p} = (0, 0, p)$, and assume $p = p_3 > 0$. The Green equation is now written as

$$\begin{aligned} &\partial_z^2 A_3 = 0, \\ &(\partial_z^2 - p^2)A_i = \delta(z - z_0)\xi_i, \quad i = 1, 2, \end{aligned} \tag{5.280}$$

and the boundary condition as

$$\partial_z A_3|_{z=0} = 0, \qquad \left(p\epsilon_{ij} A_j - \frac{\alpha}{k}\partial_z A_i\right)\Big|_{z=0} = 0, \quad i = 1, 2. \tag{5.281}$$

The z-independent part of A_3 can be gauged away. We may then take the solution

$$\begin{aligned} &A_3 = 0, \\ &A_i = \theta(z - z_0)\left[g_i(p) + h_i(p)\right] e^{-p(z-z_0)} + \theta(z_0 - z)\left[g_i(p)e^{-p(z-z_0)} + h_i(p)e^{p(z-z_0)}\right], \end{aligned} \tag{5.282}$$

where $g_i(p)$ and $h_i(p)$ obey

$$\begin{aligned} &- p(g_i + h_i) - (-pg_i + ph_i) = \xi_i. \\ &\epsilon_{ij}(g_j e^{pz_0} + h_j e^{-pz_0}) + \frac{\alpha}{k}(g_i e^{pz_0} - h_i e^{-pz_0}) = 0. \end{aligned} \tag{5.283}$$

The solutions are

$$g_i = \frac{e^{-2pz_0}}{2(1+\frac{\alpha^2}{k^2})p}\left[(1-\frac{\alpha^2}{k^2})\xi_i + 2\frac{\alpha}{k}\epsilon_{ij}\xi_j\right], \qquad h_i = -\frac{\xi_i}{2p}. \tag{5.284}$$

The nontrivial components of Green's function are thus given by

$$\begin{aligned} &G_{33} = 0, \\ &G_{ij} = \frac{1}{2p}\left[e^{-p(z+z_0)}\frac{(1-\frac{\alpha^2}{k^2})\delta_{ij} + 2\frac{\alpha}{k}\epsilon_{ij}}{1+\frac{\alpha^2}{k^2}}\right] - \frac{\delta_{ij}}{2p}\Big[\theta(z-z_0)e^{-p(z-z_0)} + \theta(z_0-z)e^{p(z-z_0)}\Big]. \end{aligned} \tag{5.285}$$

In particular, we find the change of the bulk Green's function due to the changing of the boundary condition,

$$G^{(k)}_{ij} - G^{(\infty)}_{ij} \equiv \Delta_{ij}(p,z,z_0) = \frac{\alpha}{kp}\frac{\epsilon_{ij} - \frac{\alpha}{k}\delta_{ij}}{1+\frac{\alpha^2}{k^2}}e^{-p(z+z_0)}. \tag{5.286}$$

The boundary to bulk propagator for $k=\infty$ can be obtained by taking $z_0 \to 0$ limit on $z_0^{-1}G^{(\infty)}$, giving

$$\begin{aligned} K_{33} &= 0, \\ K_{ij} &= -e^{-pz}\delta_{ij}. \end{aligned} \tag{5.287}$$

We observe that Δ_{ij} factorizes into the product of two boundary to bulk propagators, $K(p,z)$ and $K(p,z_0)$, multiplied by

$$M_{ij}(p) = \frac{\alpha}{kp}\frac{\epsilon_{ij} - \frac{\alpha}{k}\delta_{ij}}{1+\frac{\alpha^2}{k^2}}. \tag{5.288}$$

This is reminiscent of the change of scalar propagator due to boundary conditions [32, 23]. So far we worked in the special case $p = p_3$. Restoring rotational invariance, (5.288) is replaced by

$$\begin{aligned} M_{ij}(p) &= \frac{\alpha}{k|p|}\frac{\epsilon_{ijk}\frac{p^k}{|p|} - \frac{\alpha}{k}(\delta_{ij} - \frac{p_ip_j}{p^2})}{1+\frac{\alpha^2}{k^2}} \\ &= \frac{\alpha/k}{1+\alpha^2/k^2}\epsilon_{ijk}\frac{p^k}{p^2} - \frac{\alpha^2/k^2}{1+\alpha^2/k^2}\frac{\delta_{ij} - \frac{p_ip_j}{p^2}}{|p|}. \end{aligned} \tag{5.289}$$

In the boundary CFT, the change of boundary condition amounts to coupling the $U(1)$ current J^i to a boundary gauge field A_i at Chern-Simons level k. $M_{ij}(p)$ is proportional to the two-point function of A_i in the Lorentz gauge $\partial_j A^j = 0$. Namely,

$$\langle A_i(p) A_j(-q) \rangle = \frac{32}{\widetilde{N}} M_{ij}(p) (2\pi)^3 \delta^3(p-q), \tag{5.290}$$

where $\widetilde{N}$ is the overall normalization factor in the two-point function of the current J_i,

$$\langle J_i(p) J_j(-q) \rangle = -\frac{\widetilde{N}|p|}{32} \left(\delta_{ij} - \frac{p_i p_j}{p^2} \right) (2\pi)^3 \delta^3(p-q). \tag{5.291}$$

Our convention is such that in the free theory $\widetilde{N}$ counts the total number of complex scalars and fermions. Note that here we are normalizing the current coupled to the Chern-Simons gauge field according to the convention for nonabelian gauge group generators, $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$ for generators t^a, t^b in the fundamental representation. This is also the normalization convention we use to define the Chern-Simons level k (which differs by a factor of 2 from the natural convention for $U(1)$ gauge group).

To see this, note that the inverse of the matrix M_{ij} in (5.288), restricted to directions transverse to $\vec{p} = p_3 \hat{e}_3$, is

$$(M_\perp^{-1})_{ij} = \frac{kp}{\alpha} \epsilon_{ij} + \delta_{ij} p. \tag{5.292}$$

After restoring rotational invariance, this is

$$(M_\perp^{-1})_{ij} = \frac{k}{\alpha} \epsilon_{ijk} p^k + \left(\delta_{ij} - \frac{p_i p_j}{p^2} \right) |p| \tag{5.293}$$

which for $\alpha = \frac{\pi}{8}\widetilde{N}$ precisely matches $32\widetilde{N}^{-1}$ times the kinetic term of the Chern-Simons gauge field plus the contribution to the self energy of A_i from $\langle J_i(p) J_j(-p) \rangle_{CFT_\infty}$.

5.D Supersymmetric Chern-Simons vector models at large N

In this appendix, we review the Lagrangian of Chern-Simons vector models with various numbers of supersymmetries and/or superpotentials. The scalar potentials and scalar-fermion coupling resulting from the coupling to auxiliary fields in the Chern-Simons gauge multiplet and superpotentials can be expressed in terms of bosonic or fermionic singlets under the $U(N)$ Chern-Simons gauge group as double trace or triple trace terms. These can be matched with the change of boundary conditions in the holographically dual Vasiliev theories in AdS_4, described in Section 5.4.

5.D.1 $\mathcal{N}=2$ theory with M $\square$ chiral multiplets

The action of the $\mathcal{N}=2$ pure Chern-Simons theory in Lorentzian signature is

$$S_{CS}^{\mathcal{N}=2} = \frac{k}{4\pi}\int \mathrm{Tr}(A\wedge dA + \frac{2}{3}A^3 - \bar{\chi}\chi + 2D\sigma), \tag{5.294}$$

where $\chi, \bar{\chi}$ and D, σ are fermionic and bosonic auxiliary fields. The M chiral multiplets in the fundamental representation couple to the gauge multiplet through the action

$$S_m = \int \sum_{i=1}^{M} \left[D_\mu\bar{\phi}^i D^\mu\phi_i + \bar{\psi}^i(\not{D}+\sigma)\psi_i + \bar{\phi}^i(\sigma^2 - D)\phi_i + \bar{\psi}^i\bar{\chi}\phi_i + \bar{\phi}^i\chi\psi_i - \bar{F}F\right]. \tag{5.295}$$

We will focus on the matter coupling

$$\frac{k}{4\pi}\mathrm{Tr}(-\bar{\chi}\chi + 2D\sigma) + \int \sum_{i=1}^{M} \left[\bar{\psi}^i\sigma\psi_i + \bar{\phi}^i(\sigma^2 - D)\phi_i + \bar{\psi}^i\bar{\chi}\phi_i + \bar{\phi}^i\chi\psi_i - \bar{F}F\right]. \tag{5.296}$$

Integrating out the auxiliary fields, we obtain the scalar potential and scalar-fermion coupling,

$$V = \frac{4\pi^2}{k^2}\bar{\phi}^i\phi_j\bar{\phi}^j\phi_k\bar{\phi}^k\phi_i + \frac{4\pi}{k}\bar{\phi}^j\phi_i\bar{\psi}^i\psi_j + \frac{2\pi}{k}\bar{\psi}^i\phi_j\bar{\phi}^j\psi_i. \tag{5.297}$$

For the purpose of comparing with vector models of other numbers of supersymmetries, it is useful to consider the $M = 2$ case. Let us define bosonic and fermionic gauge invariant bilinears in the matter fields,

$$\Phi^a_+ = \bar{\phi}^i\phi_j(\sigma^a)^j{}_i, \quad \Phi^a_- = \bar{\psi}^i\psi_j(\sigma^a)^j{}_i, \quad \Psi^i{}_j = \bar{\phi}^i\psi_j, \tag{5.298}$$

where $\sigma^a = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3)$. The non-supersymmetric theory with two flavors and without matter self-interaction V would have had $SU(2)_b \times SU(2)_f$ flavor symmetry acting on the bosons and fermions separately. With respect to this symmetry, Φ^a_+, Φ^a_- and $\Psi^i{}_j$ are in the representation $(\mathbf{1} \oplus \mathbf{3}, \mathbf{1})$, $(\mathbf{1}, \mathbf{1} \oplus \mathbf{3})$ and $(\mathbf{2}, \mathbf{2})$ respectively. Expressed in terms of the bosonic and fermionic singlets, V can be written as

$$V = \frac{\pi^2}{2k^2}\Phi^a_+\Phi^b_+\Phi^c_+ \mathrm{Tr}\left(\sigma^a\sigma^b\sigma^c\right) + \frac{2\pi}{k}\Phi^a_+\Phi^a_- + \frac{2\pi}{k}\bar{\Psi}^i{}_j\Psi^j{}_i. \tag{5.299}$$

Note that the (fermion singlet)^2 terms is invariant under $SU(2)_b \times SU(2)_f$, whereas the (bosonic singlet)^2 term and the scalar potential explicitly break $SU(2)_b \times SU(2)_f$ to the diagonal flavor $SU(2)$.

Indeed, the boundary conditions of the conjectured holographic dual described in Section 5.4.2 are such that the fermionic boundary condition (characterized by γ) is invariant under the $SO(4) \sim SU(2)_b \times SU(2)_f$ that rotates the four Grassmannian variables of supersymmetric Vasiliev theory, while the scalar boundary condition only preserve an $SU(2) \sim SO(3)$ subgroup.

5.D.2 $\mathcal{N} = 1$ theory with M $\square$ chiral multiplets

The $\mathcal{N} = 2$ theory in the previous section admits a one-parameter family of exactly marginal deformations that preserves $\mathcal{N} = 1$ supersymmetry. The matter coupling of this $\mathcal{N} = 1$ theory is given by

$$\begin{aligned} V =& \frac{4\pi^2\omega^2}{k^2}\bar{\phi}^i\phi_j\bar{\phi}^j\phi_k\bar{\phi}^k\phi_i + \frac{2\pi(1+\omega)}{k}\bar{\phi}^j\phi_i\bar{\psi}^i\psi_j + \frac{2\pi\omega}{k}\bar{\psi}^i\phi_j\bar{\phi}^j\psi_i \\ &+ \frac{\pi(\omega-1)}{k}(\bar{\psi}^i\phi_j\bar{\psi}^j\phi_i + \bar{\phi}^i\psi_j\bar{\phi}^j\psi_i), \end{aligned} \tag{5.300}$$

where ω is a real deformation parameter. The $\mathcal{N} = 2$ theory is given by $\omega = 1$.

5.D.3 The $\mathcal{N} = 2$ theory with M $\square$ chiral multiplets and M $\overline{\square}$ chiral multiplets

Now we turn to the $\mathcal{N} = 2$ Chern-Simons vector model with an equal number M of fundamental and anti-fundamental chiral multiplets. This model differs from the $\mathcal{N} = 2$ theory with $2M$ fundamental chiral multiplets through the scalar-fermion coupling and scalar potential only. The part of the Lagrangian that couples matter fields to the auxiliary fields in the gauge multiplet is given by

$$\begin{aligned} &\frac{k}{4\pi}\mathrm{Tr}(-\bar{\chi}\chi + 2D\sigma) + \sum_{i=1}^{M}\left[\bar{\psi}^i\sigma\psi_i + \bar{\phi}^i(\sigma^2 - D)\phi_i + \bar{\psi}^i\bar{\chi}\phi_i + \bar{\phi}^i\chi\psi_i - \bar{F}F\right] \\ &+ \sum_{i=1}^{M}\left[-\tilde{\psi}^i\sigma\bar{\bar{\tilde{\psi}}}_i + \tilde{\phi}^i(\sigma^2 + D)\bar{\bar{\tilde{\phi}}}_i - \tilde{\psi}^i\chi\bar{\bar{\tilde{\phi}}}_i - \tilde{\phi}^i\bar{\chi}\bar{\bar{\tilde{\psi}}}_i - \tilde{F}\bar{\bar{\tilde{F}}}\right]. \end{aligned} \tag{5.301}$$

Integrating out the auxiliary fields, we obtain

$$\begin{aligned} V_d =& \frac{4\pi^2}{k^2}(\bar{\phi}^k\phi_i\bar{\phi}^i\phi_j\bar{\phi}^j\phi_k - \bar{\phi}^k\bar{\bar{\tilde{\phi}}}_i\tilde{\phi}^i\bar{\bar{\tilde{\phi}}}_j\tilde{\phi}^j\phi_k - \bar{\phi}^k\bar{\bar{\tilde{\phi}}}_i\tilde{\phi}^i\phi_j\bar{\phi}^j\phi_k + \tilde{\phi}^k\bar{\bar{\tilde{\phi}}}_i\tilde{\phi}^i\bar{\bar{\tilde{\phi}}}_j\tilde{\phi}^j\bar{\bar{\tilde{\phi}}}_k) \\ &+ \frac{4\pi}{k}(\bar{\phi}^j\phi_i\bar{\psi}^i\psi_j - \bar{\phi}^j\bar{\bar{\tilde{\psi}}}_i\tilde{\phi}^i\psi_j - \tilde{\psi}^j\phi_i\bar{\psi}^i\bar{\bar{\tilde{\phi}}}_j + \tilde{\psi}^i\bar{\bar{\tilde{\psi}}}_j\tilde{\phi}^j\bar{\bar{\tilde{\phi}}}_i) \\ &+ \frac{2\pi}{k}(\bar{\psi}^i\phi_j\bar{\phi}^j\psi_i - \bar{\psi}^i\bar{\bar{\tilde{\phi}}}_j\tilde{\phi}^j\psi_i - \tilde{\psi}^i\phi_j\bar{\phi}^j\bar{\bar{\tilde{\psi}}}_i + \tilde{\psi}^i\bar{\bar{\tilde{\phi}}}_j\tilde{\phi}^j\bar{\bar{\tilde{\psi}}}_i). \end{aligned} \tag{5.302}$$

5.D.4 The $\mathcal{N}=3$ theory with M hypermultiplets

The $\mathcal{N}=3$ Chern-Simons vector model with M hypermultiplets can be obtained from the $\mathcal{N}=2$ theory described in the previous subsection by adding the superpotential [89, 72]

$$W=-\frac{k}{8\pi}\operatorname{tr}\varphi^2+\tilde{\Phi}^i\varphi\Phi_i \tag{5.303}$$

where φ is an auxiliary $\mathcal{N}=2$ chiral superfield. Integrating out φ, we obtain a quartic superpotential

$$W=\frac{2\pi}{k}(\tilde{\Phi}^i\Phi_j)(\tilde{\Phi}^j\Phi_i). \tag{5.304}$$

After integrating over the superspace, we obtain

$$\int d^2\theta\; W+c.c.=\frac{2\pi}{k}\left[2\tilde{\phi}^i\phi_j(\tilde{\phi}^jF_i+\tilde{F}^j\phi_i+\tilde{\psi}^j\psi_i)+(\tilde{\psi}^i\phi_j+\tilde{\phi}^i\psi_j)(\tilde{\psi}^j\phi_i+\tilde{\phi}^j\psi_i)+c.c.\right]. \tag{5.305}$$

Integrating out the auxiliary fields $F,\tilde{F}$, the W-term potential is

$$\begin{aligned}V_w=&\frac{2\pi}{k}\left[2(\tilde{\phi}^i\phi_j)(\tilde{\psi}^j\psi_i)+(\tilde{\psi}^i\phi_j+\tilde{\phi}^i\psi_j)(\tilde{\psi}^j\phi_i+\tilde{\phi}^j\psi_i)+c.c\right]\\&+\frac{16\pi^2}{k^2}(\bar{\phi}^j\bar{\tilde{\phi}}_i)(\tilde{\phi}^i\phi_k)(\tilde{\phi}^k\bar{\tilde{\phi}}_j)+\frac{16\pi^2}{k^2}(\bar{\phi}^j\bar{\tilde{\phi}}_i)(\bar{\phi}^i\phi_k)(\tilde{\phi}^k\phi_j).\end{aligned} \tag{5.306}$$

The total potential is given by the D-term plus W-term potentials:

$$V=V_d+V_w. \tag{5.307}$$

To make the $SO(3)$ R-symmetry manifest, we rewrite the potential in terms of the $SO(3)$ doublets:

$$(\phi_i^A)=\begin{pmatrix}\phi_i\\ \bar{\tilde{\phi}}_i\end{pmatrix},\quad(\psi_{A,i})=\begin{pmatrix}\psi_i\\ \bar{\tilde{\psi}}_i\end{pmatrix}. \tag{5.308}$$

The D-term and W-term potentials are

$$\begin{aligned}V_d =&\frac{4\pi^2}{k^2}\left[(\bar\phi_1\phi^1)(\bar\phi_1\phi^1)(\bar\phi_1\phi^1)-(\bar\phi_1\phi^2)(\bar\phi_2\phi^2)(\bar\phi_2\phi^1)-(\bar\phi_1\phi^2)(\bar\phi_2\phi^1)(\bar\phi_1\phi^1)+(\bar\phi_2\phi^2)(\bar\phi_2\phi^2)(\bar\phi_2\phi^2)\right]\\ &+\frac{4\pi}{k}\left[(\bar\phi_1\phi^1)(\bar\psi^1\psi_1)-(\bar\phi_1\psi_2)(\bar\phi_2\psi_1)-(\bar\psi^2\phi^1)(\bar\psi^1\phi^2)+(\bar\psi^2\psi_2)(\bar\phi_2\phi^2)\right]\\ &+\frac{2\pi}{k}\left[(\bar\psi^1\phi^1)(\bar\phi_1\psi_1)-(\bar\psi^1\phi^2)(\bar\phi_2\psi_1)-(\bar\psi^2\phi^1)(\bar\phi_1\psi_2)+(\bar\psi^2\phi^2)(\bar\phi_2\psi_2)\right],\end{aligned} \tag{5.309}$$

and

$$\begin{aligned}V_w =&\frac{2\pi}{k}\left[2(\bar\phi_2\phi^1)(\bar\psi^2\psi_1)+(\bar\psi^2\phi^1+\bar\phi_2\psi_1)(\bar\psi^2\phi^1+\bar\phi_2\psi_1)+c.c\right]\\ &+\frac{16\pi^2}{k^2}(\bar\phi_1\phi^2)(\bar\phi_2\phi^1)(\bar\phi_2\phi^2)+\frac{16\pi^2}{k^2}(\bar\phi_1\phi^2)(\bar\phi_1\phi^1)(\bar\phi_2\phi^1).\end{aligned} \tag{5.310}$$

We have also suppressed the flavor indices. The total potential can be written in a $SO(3)$ R-symmetry manifest way:

$$V = V_1 + V_2 + V_3, \tag{5.311}$$

where V_1 contains the double trace operator of the form (bosonic singlet)^2,

$$V_1 = \frac{4\pi}{k}(\bar\phi_A\phi^B)(\bar\psi^A\psi_B), \tag{5.312}$$

V_2 is the scalar potential in the form of a triple trace term,

$$V_2 = \frac{16\pi^2}{3k^2}(\bar\phi_A\phi^B)(\bar\phi_B\phi^C)(\bar\phi_C\phi^A) - \frac{4\pi^2}{3k^2}(\bar\phi_B\phi^C)(\bar\phi_A\phi^B)(\bar\phi_C\phi^A), \tag{5.313}$$

V_3 is the double trace term of the form (fermionic singlet)^2,

$$V_3 = -\frac{2\pi}{k}(\bar\psi^A\phi_B)(\bar\phi^B\psi_A)+\frac{4\pi}{k}(\bar\psi^A\phi_A)(\bar\phi^B\psi_B)+\frac{2\pi}{k}(\bar\psi^A\phi_A)(\bar\psi^B\phi_B)+\frac{2\pi}{k}(\bar\phi^A\psi_A)(\bar\phi^B\psi_B), \tag{5.314}$$

where ϕ_A, ψ^A are defined as

$$\phi_A = \phi^B\epsilon_{BA}, \quad \psi^A = \epsilon^{AB}\psi_B, \tag{5.315}$$

and $\epsilon^{AB}, \epsilon_{AB}$ are antisymmetric tensors with $\epsilon_{12} = \epsilon^{12} = 1$.

For reference in main text we will record the double trace part of the potential in $SO(3)$ vector notation. Let us define

$$\begin{aligned}
\Phi_+^a &= \bar{\phi}_A \phi^B (\sigma^a)_B^{\ A} \quad &\Leftrightarrow \quad \bar{\phi}_A \phi^B &= \frac{1}{2} \Phi_+^a (\bar{\sigma}^a)_A^{\ B} \\
\Phi_-^a &= \bar{\psi}^A \psi_B (\sigma^a)_A^{\ B} \quad &\Leftrightarrow \quad \bar{\psi}^A \psi_B &= \frac{1}{2} \Phi_-^a (\bar{\sigma}^a)_B^{\ A} \\
\Psi^a &= \bar{\phi}_A \psi_B (\epsilon \sigma^a)^{AB} \quad &\Leftrightarrow \quad \bar{\phi}_A \psi_B &= -\frac{1}{2} \Psi^a (\sigma^a \epsilon)_{AB} \\
\bar{\Psi}^a &= -\bar{\psi}^A \phi^B (\sigma^a \epsilon)_{AB} \quad &\Leftrightarrow \quad \bar{\psi}^A \phi^B &= \frac{1}{2} \bar{\Psi}^a (\epsilon \bar{\sigma}^a)^{AB}
\end{aligned} \tag{5.316}$$

where

$$(\sigma^a)_A^{\ B} = (\sigma^i, iI)_A^{\ B}, \quad (\bar{\sigma}^a)_A^{\ B} = (\epsilon (\sigma^a)^T \epsilon)_A^{\ B} = (\sigma^a, -iI)_A^{\ B}, \quad \epsilon^{12} = \epsilon_{12} = 1.$$

Here σ^i are Pauli sigma matrices. The a,b indices runs over 1,2,3,0. A,B runs over 1,2. Ψ^a and $\bar{\Psi}^a$ transform under the as vectors of $SO(4)$ which under $SO(3)$ transform as singlet(a=0) and a vector(a=1,2,3) while ϕ^A, ψ_A transform as doublets of $SU(2)$.

$$\begin{aligned}
V_1 =& \frac{2\pi}{k} \Phi_+^a \Phi_-^b \eta_{ab}, \\
V_3 =& \frac{2\pi}{k} \left(\frac{1}{2} \bar{\Psi}^a \Psi^b \delta^{ab} - 2\bar{\Psi}^0 \Psi^0 - \bar{\Psi}^0 \bar{\Psi}^0 - \Psi^0 \Psi^0 \right).
\end{aligned} \tag{5.317}$$

5.D.5 A family of $\mathcal{N} = 2$ theories with a $\square$ chiral multiplet and a $\overline{\square}$ chiral multiplet

If we deformed the superpotential in the above subsection as

$$W = \frac{2\pi\omega}{k} (\tilde{\Phi}^i \Phi_j)(\tilde{\Phi}^j \Phi_i), \tag{5.318}$$

the $\mathcal{N} = 3$ supersymmetry is broken to $\mathcal{N} = 2$. In this case, the potential is

$$V = V_1 + V_2 + V_3, \tag{5.319}$$

where V_1 contains the double trace operator of the form (bosonic singlet)^2,

$$V_1 = \frac{4\pi}{k}\left[(\bar\phi_1\phi^1)(\bar\psi^1\psi_1) + (\bar\phi_2\phi^2)(\bar\psi^2\psi_2) + \omega(\bar\phi_2\phi^1)(\bar\psi^2\psi_1) + \omega(\bar\phi_1\phi^2)(\bar\psi^1\psi_2)\right], \tag{5.320}$$

V_2 is the scalar potential in the form of a triple trace term,

$$\begin{aligned} V_2 =& \frac{4\pi^2}{k^2}\left[(\bar\phi_1\phi^1)(\bar\phi_1\phi^1)(\bar\phi_1\phi^1) - (\bar\phi_2\phi^1)(\bar\phi_1\phi^2)(\bar\phi_2\phi^2) - (\bar\phi_1\phi^2)(\bar\phi_2\phi^1)(\bar\phi_1\phi^1) + (\bar\phi_2\phi^2)(\bar\phi_2\phi^2)(\bar\phi_2\phi^2)\right] \\ &+ \frac{16\pi^2\omega}{k^2}(\bar\phi_1\phi^2)(\bar\phi_2\phi^1)(\bar\phi_2\phi^2) + \frac{16\pi^2\omega}{k^2}(\bar\phi_1\phi^2)(\bar\phi_1\phi^1)(\bar\phi_2\phi^1), \end{aligned} \tag{5.321}$$

V_3 is the double trace term of the form (fermionic singlet)^2,

$$\begin{aligned} V_3 =& \frac{2\pi}{k}\left[(\bar\psi^1\phi^1)(\bar\phi_1\psi_1) - (\bar\psi^1\phi^2)(\bar\phi_2\psi_1) - (\bar\psi^2\phi^1)(\bar\phi_1\psi_2) + (\bar\psi^2\phi^2)(\bar\phi_2\psi_2)\right] \\ &+ \frac{4\pi}{k}\left[-(\bar\phi_1\psi_2)(\bar\phi_2\psi_1) - (\bar\psi^2\phi^1)(\bar\psi^1\phi^2)\right] + \frac{2\pi\omega}{k}\left[(\bar\psi^2\phi^1)(\bar\psi^2\phi^1) + 2(\bar\phi_2\psi_1)(\bar\psi^2\phi^1) + (\bar\phi_2\psi_1)(\bar\phi_2\psi_1)\right. \\ &\left.+(\bar\phi_1\psi_2)(\bar\phi_1\psi_2) + 2(\bar\psi^1\phi^2)(\bar\phi_1\psi_2) + (\bar\psi^1\phi^2)(\bar\psi^1\phi^2)\right]. \end{aligned} \tag{5.322}$$

5.D.6 The $\mathcal{N} = 4$ theory with one hypermultiplet

As shown by [73], $\mathcal{N} = 3$ $U(N)_k$ Chern-Simons vector model with M hypermultiplets can be deformed to an $\mathcal{N} = 4$ quiver type Chern-Simons matter theory by gauging (a subgroup of) the flavor group with another $\mathcal{N} = 3$ Chern-Simons gauge multiplet, at the opposite level $-k$. Here we will focus on the case where the entire $U(M)$ is gauged, so that the resulting $\mathcal{N} = 4$ theory has $U(N)_k \times U(M)_{-k}$ Chern-Simons gauge group and a single bifundamental hypermultiplet. This $\mathcal{N} = 4$ theory will still be referred to as a vector model, as we will be thinking of the 't Hooft limit of taking N, k large and M kept finite. As we have seen, turning on the finite Chern-Simons level for the flavor group $U(M)$ amounts to simply changing the boundary condition on the $U(M)$ vector gauge fields in the bulk Vasiliev theory.

The part of the Lagrangian that couples matter fields to the auxiliary fields in the gauge multiplet is given by

$$\begin{aligned}
&\frac{k}{4\pi}\mathrm{Tr}(-\bar{\chi}\chi+2D\sigma)-\frac{k}{4\pi}\mathrm{Tr}(-\bar{\hat{\chi}}\hat{\chi}+2\hat{D}\hat{\sigma})\\
&+\left[\bar{\psi}\sigma\psi+\bar{\phi}(\sigma^2-D)\phi+\bar{\psi}\bar{\chi}\phi+\bar{\phi}\chi\psi-\hat{\sigma}\bar{\psi}\psi+\left(\hat{\sigma}^2+\hat{D}\right)\bar{\phi}\phi-\bar{\psi}\phi\bar{\hat{\chi}}-\hat{\chi}\bar{\phi}\psi-2\hat{\sigma}\bar{\phi}\sigma\phi-\bar{F}F\right]\\
&+\left[-\tilde{\psi}\sigma\bar{\tilde{\psi}}+\tilde{\phi}(\sigma^2+D)\bar{\tilde{\phi}}-\tilde{\psi}\chi\bar{\tilde{\phi}}-\tilde{\phi}\bar{\chi}\bar{\tilde{\psi}}+\hat{\sigma}\tilde{\psi}\bar{\tilde{\psi}}+\left(\hat{\sigma}^2-\hat{D}\right)\tilde{\phi}\bar{\tilde{\phi}}+\bar{\hat{\chi}}\tilde{\phi}\bar{\tilde{\psi}}+\tilde{\psi}\bar{\tilde{\phi}}\hat{\chi}-2\hat{\sigma}\tilde{\phi}\sigma\bar{\tilde{\phi}}-\tilde{F}\bar{\tilde{F}}\right],
\end{aligned} \tag{5.323}$$

where we suppressed the both $SU(N)$ and $SU(M)$ indices. Integrating out the auxiliary fields, we obtain the potential:

$$\begin{aligned}
V=&\frac{2\pi}{k}\bar{\phi}_A\phi^A\bar{\psi}^B\psi_B+\frac{4\pi^2}{3k^2}(\bar{\phi}_A\phi^B\bar{\phi}_B\phi^C\bar{\phi}_C\phi^A+\bar{\phi}_A\phi^A\bar{\phi}_B\phi^B\bar{\phi}_C\phi^C-2\bar{\phi}_B\phi^C\bar{\phi}_A\phi^B\bar{\phi}_C\phi^A)\\
&+\frac{2\pi}{k}\left(-\bar{\psi}^A\phi^B\bar{\phi}_B\psi_A+\bar{\phi}^A\psi^B\bar{\phi}_A\psi_B+\bar{\psi}^A\phi^B\bar{\psi}_A\phi_B\right).
\end{aligned} \tag{5.324}$$

The complex scalar ϕ^A and the fermion ψ_A transform as $(2,1)$ and $(1,2)$ under the $SO(4)=SU(2)\times SU(2)$ R-symmetry. The potential (5.324) is manifestly invariant under the R-symmetry.

For reference to main text we now record the double trace part of this potential in $SO(4)$ vector notation. Using the definitions (5.316), the (scalar singlet)^2 part(V_1) and (fermion singlet)^2 part(V_3) are given by

$$\begin{aligned}
V_1&=-\frac{2\pi}{k}\Phi^0_+\Phi^0_-,\\
V_2&=-\frac{\pi}{k}\left(\bar{\Psi}^a\Psi^a+\bar{\Psi}^a\bar{\Psi}^a+\Psi^a\Psi^a\right).
\end{aligned} \tag{5.325}$$

5.D.7 $\mathcal{N}=3$ $U(N_{k_1})\times U(M)_{k_2}$ theories with one hypermultiplet

The $\mathcal{N}=4$ theory in the previous section sits in a discrete one parameter family of $\mathcal{N}=3$ $U(N)_{k_1}\times U(M)_{k_2}$ theories with one hypermultiplet. The potential can be written in an $SO(3)$ R-symmetry manifest way:

$$V = V_1 + V_2 + V_3, \tag{5.326}$$

where V_1 contains the double trace operator of the form (bosonic singlet)^2,

$$V_1 = \frac{4\pi}{k_1}\bar{\phi}_A\phi^B\bar{\psi}^A\psi_B + \frac{2\pi}{k_2}\left[\bar{\phi}_A\phi^A\bar{\psi}_B\psi^B + 2\bar{\phi}_A\phi^B\bar{\psi}^A\psi_B\right], \tag{5.327}$$

V_2 is the scalar potential in the form of triple trace term. V_3 is the double trace term of the form (fermionic singlet)^2,

$$\begin{aligned} V_3 =& \frac{2\pi}{k_1}\left[-\bar{\psi}^A\phi_B\bar{\phi}^B\psi_A + 2\bar{\psi}^A\phi_A\bar{\phi}^B\psi_B + \bar{\psi}^A\phi_A\bar{\psi}^B\phi_B + \bar{\phi}^A\psi_A\bar{\phi}^B\psi_B\right] \\ &+ \frac{2\pi}{k_2}\left[2\bar{\psi}^A\phi^B\bar{\phi}_A\psi_B + \bar{\psi}^A\phi_B\bar{\psi}^B\phi_A + \bar{\phi}_A\psi^B\bar{\phi}_B\psi^A\right]. \end{aligned} \tag{5.328}$$

In the notation defined in (5.316) V_1 and V_3 becomes

$$\begin{aligned} V_1 =& \frac{2\pi}{k_1}\Phi_+^a\Phi_-^b\eta_{ab} + \frac{2\pi}{k_2}\left(\Phi_+^0\Phi_-^0 + \Phi_+^a\Phi_-^b\eta_{ab}\right), \\ V_3 =& \frac{2\pi}{k_1}\left(\frac{1}{2}\bar{\Psi}^a\Psi^b\delta^{ab} - 2\bar{\Psi}^0\Psi^0 - \bar{\Psi}^0\bar{\Psi}^0 - \Psi^0\Psi^0\right) + \frac{2\pi}{k_2}\left(\bar{\Psi}^a\Psi^b\eta^{ab} + \frac{1}{2}\bar{\Psi}^a\bar{\Psi}^b\eta_{ab} + \frac{1}{2}\Psi^a\Psi^b\eta^{ab}\right). \end{aligned} \tag{5.329}$$

5.D.8 The $\mathcal{N} = 6$ theory

The above $\mathcal{N} = 4$ theory can be generalized to a quiver $\mathcal{N} = 3$ theory with $\tilde{n}$ hypermultiplets by starting with the $\mathcal{N} = 3$ $U(N)_k$ Chern-Simons vector model with $\tilde{n}M$ hypermultiplets and only gauging the $U(M)$ subgroup, of the $U(\tilde{n}M)$ flavor group, at level $-k$ with another $\mathcal{N} = 3$ Chern-Simons gauge multiplet. The resulting theory has $SU(\tilde{n})$ flavor symmetry. For generic value of $\tilde{n}$, the theory has $\mathcal{N} = 3$ sypersymmetry, but for $\tilde{n} = 1, 2$, the theory exhibits $\mathcal{N} = 4, 6$ sypersymmetry, respectively. Let us focus on the

$\tilde{n} = 2$ case. The part of the Lagrangian that couples matter fields to the auxiliary fields in the gauge multiplet is given by

$$\begin{aligned}
&\frac{k}{4\pi}\mathrm{Tr}(-\bar{\chi}\chi + 2D\sigma) - \frac{k}{4\pi}\mathrm{Tr}(-\bar{\hat{\chi}}\hat{\chi} + 2\hat{D}\hat{\sigma}) \\
&+ \Big[\bar{\psi}_a\sigma\psi^a + \bar{\phi}_a(\sigma^2 - D)\phi^a + \bar{\psi}_a\bar{\chi}\phi^a + \bar{\phi}_a\chi\psi^a - \hat{\sigma}\bar{\psi}_a\psi^a \\
&\quad + \left(\hat{\sigma}^2 + \hat{D}\right)\bar{\phi}_a\phi^a - \bar{\psi}_a\phi^a\bar{\hat{\chi}} - \hat{\chi}\bar{\phi}_a\psi^a - 2\hat{\sigma}\bar{\phi}_a\sigma\phi^a - \bar{F}_aF^a\Big] \\
&+ \Big[-\tilde{\psi}_{\dot{a}}\sigma\bar{\tilde{\psi}}^{\dot{a}} + \tilde{\phi}_{\dot{a}}(\sigma^2 + D)\bar{\tilde{\phi}}^{\dot{a}} - \tilde{\psi}_{\dot{a}}\chi\bar{\tilde{\phi}}^{\dot{a}} - \tilde{\phi}_{\dot{a}}\bar{\chi}\bar{\tilde{\psi}}^{\dot{a}} + \hat{\sigma}\tilde{\psi}_{\dot{a}}\bar{\tilde{\psi}}^{\dot{a}} \\
&\quad + \left(\hat{\sigma}^2 - \hat{D}\right)\tilde{\phi}_{\dot{a}}\bar{\tilde{\phi}}^{\dot{a}} + \bar{\hat{\chi}}\tilde{\phi}_{\dot{a}}\bar{\tilde{\psi}}^{\dot{a}} + \tilde{\psi}_{\dot{a}}\bar{\tilde{\phi}}^{\dot{a}}\hat{\chi} - 2\hat{\sigma}\tilde{\phi}_{\dot{a}}\sigma\bar{\tilde{\phi}}^{\dot{a}} - \tilde{F}_{\dot{a}}\bar{\tilde{F}}^{\dot{a}}\Big],
\end{aligned} \tag{5.330}$$

where $a, \dot{a} = 1, 2$ are the $SU(2) \times SU(2)$ indices. There is also an superpotential

$$W = -\frac{2\pi}{k}\mathrm{Tr}\,(\tilde{\Phi}^{\dot{a}}\Phi^b\tilde{\Phi}_{\dot{a}}\Phi_b). \tag{5.331}$$

After integrating over the superspace, we obtain

$$\int d^2\theta\ W + c.c. = -\frac{2\pi}{k}\left[2\tilde{\phi}^{\dot{a}}\phi^b(\tilde{\phi}_{\dot{a}}F_b + \tilde{F}_{\dot{a}}\phi_b + \tilde{\psi}_{\dot{a}}\psi_b) + (\tilde{\psi}^{\dot{a}}\phi^b + \tilde{\phi}^{\dot{a}}\psi^b)(\tilde{\psi}_{\dot{a}}\phi_b + \tilde{\phi}_{\dot{a}}\psi_b) + c.c.\right]. \tag{5.332}$$

After integrating out all the auxiliary fields, the resulting potential can be written in a $SO(6)$ R-symmetry manifest way:

$$V = V_1 + V_2 + V_3, \tag{5.333}$$

where V_1 contains the double trace operator of the form (bosonic singlet)^2,

$$\begin{aligned}
V_1 =& -\frac{2\pi}{k}(\bar{\phi}_{1a}\phi^{1a}\bar{\psi}^{2\dot{b}}\psi_{2\dot{b}} + \bar{\phi}_{1a}\phi^{1a}\bar{\psi}^{1b}\psi_{1b} + \bar{\phi}_{2\dot{a}}\phi^{2\dot{a}}\bar{\psi}^{2\dot{b}}\psi_{2\dot{b}} + \bar{\phi}_{2\dot{a}}\phi^{2\dot{a}}\bar{\psi}^{1b}\psi_{1b}) \\
&+ \frac{4\pi}{k}(\bar{\phi}_{2\dot{a}}\phi^{1b}\bar{\psi}^{2\dot{a}}\psi_{1b} + \bar{\phi}_{1b}\phi^{2\dot{a}}\bar{\psi}^{1b}\psi_{2\dot{a}} + \bar{\phi}_{1a}\phi^{1b}\bar{\psi}^{1a}\psi_{1b} + \bar{\phi}_{2\dot{a}}\phi^{2\dot{b}}\bar{\psi}^{2\dot{a}}\psi_{2\dot{b}}) \\
=& -\frac{2\pi}{k}\bar{\phi}_A\phi^A\bar{\psi}^B\psi_B + \frac{4\pi}{k}\bar{\phi}_A\phi^B\bar{\psi}^A\psi_B
\end{aligned} \tag{5.334}$$

where we have rewrite the potential in terms of the $SO(3)$ doublets (5.308), and $A, B = (11, 12, 21, 22)$ are the $SO(6)$ spinor indices. V_2 is the scalar potential in the form of triple trace term. V_3 is the double trace term of the form (fermionic singlet)^2,

$$V_3 = \frac{2\pi}{k}\left(\bar{\psi}^A\phi^B\bar{\phi}_B\psi_A - 2\bar{\psi}^A\phi^B\bar{\phi}_A\psi_B\right) + \frac{2\pi}{k}(\epsilon_{ABCD}\bar{\psi}^A\phi^B\bar{\psi}^C\phi^D + \epsilon^{ABCD}\bar{\phi}_A\psi_B\bar{\phi}_C\psi_D) \tag{5.335}$$

where $\epsilon_{11,12,21,22} = \epsilon^{11,12,21,22} = 1$.

5.D.9 $\mathcal{N} = 3$ $U(N)_{k_1} \times U(M)_{k_2}$ theories with two hypermultiplets

The $\mathcal{N} = 6$ theory in the previous section sits in a discrete one parameter family of $\mathcal{N} = 3$ $U(N)_{k_1} \times U(M)_{k_2}$ theories with two hypermultiplets. The superpotential of these theories are

$$W = \frac{2\pi}{k_1}\mathrm{Tr}\,(\tilde{\Phi}^a\Phi_b\tilde{\Phi}^b\Phi_a) + \frac{2\pi}{k_2}\mathrm{Tr}\,(\tilde{\Phi}^a\Phi_a\tilde{\Phi}^b\Phi_b), \tag{5.336}$$

where $a, b = 1, 2$ are the $SU(2)$ flavor indices. The potential can be written in an $SO(3)$ R-symmetry and $SU(2)$ flavor symmetry manifest way:

$$V = V_1 + V_2 + V_3, \tag{5.337}$$

where V_1 contains the double trace operator of the form (bosonic singlet)^2,

$$V_1 = \frac{4\pi}{k_1}\bar{\phi}_{Aa}\phi^{Bb}\bar{\psi}_b^A\psi_B^a + \frac{2\pi}{k_2}(\bar{\phi}_{Aa}\phi^{Aa}\bar{\psi}_{Bb}\psi^{Bb} + 2\bar{\phi}_{Aa}\phi^{Ba}\bar{\psi}_b^A\psi_B^b) \tag{5.338}$$

where we have rewrite the potential in terms of the $SO(3)$ doublets (5.308), and $A, B = 1, 2$ are the $SO(3)_R$ spinor indices. V_2 is the scalar potential in the form of triple trace term. V_3

is the double trace term of the form (fermionic singlet)^2,

$$V_3 = \frac{2\pi}{k_1}(\bar{\psi}^{Aa}\phi^{Bb}\bar{\phi}_{Bb}\psi_{Aa} - 2\bar{\psi}^{Aa}\phi^{Bb}\bar{\phi}_{Ab}\psi_{Ba}) + \frac{2\pi}{k_1}\epsilon_{AB}\epsilon_{CD}\bar{\psi}^A_a\phi^{Bb}\bar{\psi}^C_b\phi^{Da} + \frac{2\pi}{k_1}\epsilon^{AB}\epsilon^{CD}\bar{\phi}_{Aa}\psi^b_B\bar{\phi}_{Cb}\psi^a_D$$
$$+ \frac{4\pi}{k_2}\bar{\psi}^A_a\phi^{Ba}\bar{\phi}_{Ab}\psi^b_B + \frac{2\pi}{k_2}\epsilon_{AD}\epsilon_{CB}\bar{\psi}^A_a\phi^{Ba}\bar{\psi}^C_b\phi^{Db} + \frac{2\pi}{k_2}\epsilon^{AD}\epsilon^{CB}\bar{\phi}^a_A\psi_{aB}\bar{\phi}^b_C\psi_{Db}. \quad (5.339)$$

Now we record the double trace parts of the potential in vector notation of $SO(3)_R \times SU(2)_{flavor}$ symmetry. Let us define

$$\begin{aligned}
\Phi^{Ii}_+ &= \bar{\phi}_{Aa}\phi^{Bb}(\sigma^I)^A{}_B(\sigma^i)^a{}_b \quad &\Leftrightarrow \quad \bar{\phi}_{Aa}\phi^{Bb} &= \frac{1}{4}\Phi^{Ii}_+(\sigma^I)^B{}_A(\sigma^i)^b{}_a \\
\Phi^{Ii}_- &= \bar{\psi}^A_a\psi^b_B(\sigma^I)^B{}_A(\sigma^i)^a{}_b \quad &\Leftrightarrow \quad \bar{\psi}^A_a\psi^b_B &= \frac{1}{4}\Phi^{Ii}_-(\sigma^I)^A{}_B(\sigma^i)^b{}_a \\
\Psi^{Ii} &= \bar{\phi}_{Aa}\psi^b_B(\sigma^I\epsilon)^{AB}(\sigma^i)^a{}_b \quad &\Leftrightarrow \quad \bar{\phi}_{Aa}\psi^b_B &= -\frac{1}{4}\Psi^{Ii}(\epsilon\sigma^I)_{AB}(\bar{\sigma}^i)^b{}_a \\
\bar{\Psi}^{Ii} &= -\bar{\psi}^A_a\phi^{Bb}(\epsilon\bar{\sigma}^I)_{AB}(\bar{\sigma}^i)^a{}_b \quad &\Leftrightarrow \quad \bar{\psi}^A_a\phi^{Bb} &= -\frac{1}{4}\bar{\Psi}^{Ii}(\bar{\sigma}^I\epsilon)_{AB}(\sigma^i)^b{}_a
\end{aligned} \quad (5.340)$$

Here both set of indices I,J as well i,j run over 1,2,3,0. I,J are the vector indices of $SO(3)_R$ while i,j are vector indices of $SU(2)_{flavor}$. The 0 component corresponds to the singlet while 1,2,3 represents the vector part. In this notation the double trace potential part of the becomes

$$\begin{aligned}
V_1 =& \frac{\pi}{k_1}\Phi^{Ii}_+\Phi^{Jj}_-\eta^{IJ}\eta_{ij} - \frac{2\pi}{k_2}\Phi^{I0}_+\Phi^{J0}_-\eta^{IJ}, \\
V_3 =& \frac{2\pi}{k_1}\left(-\frac{1}{4}\bar{\Psi}^{Ii}\Psi^{Jj}\delta^{IJ}\delta^{ij} + \frac{1}{2}\bar{\Psi}^{Ii}\Psi^{Jj}\eta^{IJ}\delta^{ij} + \frac{1}{2}\left(\bar{\Psi}^{0i}\bar{\Psi}^{0j}\eta_{ij} + \Psi^{0i}\Psi^{0j}\eta_{ij}\right)\right) \\
&+ \frac{2\pi}{k_2}\left(\bar{\Psi}^{I0}\Psi^{J0}\eta^{IJ} + \frac{1}{2}\bar{\Psi}^{I0}\bar{\Psi}^{J0}\eta^{IJ} + \frac{1}{2}\Psi^{I0}\Psi^{J0}\eta^{IJ}\right).
\end{aligned} \quad (5.341)$$

The double potentials for $\mathcal{N} = 6$ theory is obtained from (5.341) on setting $k_2 = -k_1 = -k$.

5.E Argument for a Fermionic double trace shift

In this Appendix compare the boundary conditions and Lagrangian for the fixed line of $\mathcal{N} = 1$ theories to argue for the effective shift of fermionic boundary conditions induced by the Chern Simons term.

Let us use the notation $\bar{\phi}\psi = \Psi$ and $\bar{\psi}\phi = \bar{\Psi}$ for field theory single trace operators. We know that a double trace deformation proportional to $(\Psi + \bar{\Psi})^2$ is dual to fermion boundary condition (5.98) with $\alpha \propto P_{\psi_1}$. On the other hand the double trace deformation $(i\Psi - i\bar{\Psi})^2$ is dual to the fermion boundary condition with $\alpha \propto P_{\psi_2}$. Now in the zero potential theory ($w = -1$) the relevant terms in (5.300) are

$$-\frac{2\pi}{k}\left(\Psi\Psi + \bar{\Psi}\bar{\Psi} + \Psi\bar{\Psi}\right),$$

while $\alpha = \theta_0 P_{\psi_2}$. At the $\mathcal{N} = 2$ point, on the other hand, the fermion double trace term is

$$+\frac{2\pi}{k}\Psi\bar{\Psi}$$

while $\alpha = \theta_0(P_{\psi_1} + P_{\psi_2})$. Subtracting these two data points we conclude that the double trace deformation by

$$\frac{2\pi}{k}\left(\Psi + \bar{\Psi}\right)^2$$

is dual to a boundary condition deformation with $\alpha = \theta_0 P_{\psi_1}$. By symmetry it must also be that the double trace deformation by

$$-\frac{2\pi}{k}\left(\Psi - \bar{\Psi}\right)^2$$

is dual to a boundary condition deformation with $\alpha = \theta_0 P_{\psi_2}$. Adding these together, it follows that a double trace deformation by

$$\frac{8\pi}{k}\bar{\Psi}\Psi$$

is dual to the boundary condition deformation with $\alpha = \theta_0(P_{\psi_1} + P_{\psi_2})$. But the $\mathcal{N} = 2$ theory with this boundary condition has a double trace potential equal only to

$$\frac{2\pi}{k}\bar{\Psi}\Psi.$$

For consistency, it must be that the Chern Simons interaction itself induces a change in fermion boundary conditions equal to that one would have obtained from a double trace deformation

$$-\frac{6\pi}{k}\bar{\Psi}\Psi. \tag{5.342}$$

5.F Two-point functions in free field theory

Consider the action for free $SU(N)$ theory of a boson and a fermion in the fundamental representation, in flat 3 dimensional euclidean space

$$S = \int d^3x \left(\partial_\mu\bar{\phi}\partial_\mu\phi + \bar{\psi}\sigma^\mu\partial_\mu\psi\right) \tag{5.343}$$

where the $SU(N)$ in indices are suppressed and will continue to be in what follows. The Green's functions for the scalar and fermions are given by

$$\begin{aligned} G_s(x) &= \langle\bar{\phi}(x)\phi(0)\rangle = \frac{1}{4\pi|x|} \\ G_f(x) &= \langle\bar{\psi}(x)\psi(0)\rangle = \frac{x.\sigma}{4\pi|x|^3} \end{aligned} \tag{5.344}$$

Let us define the 'Single Trace' operators

$$\Phi_+ = \bar{\phi}\phi, \quad \Phi_- = \bar{\psi}\psi, \quad \Psi = \bar{\phi}\psi, \quad \bar{\Psi} = \bar{\psi}\phi, \quad J_B^\mu = i\bar{\phi}\partial^\mu\phi - \partial^\mu\bar{\phi}\phi, \quad J_F^\mu = i\bar{\psi}\sigma^\mu\psi. \tag{5.345}$$

In the free theory

$$\begin{aligned}
\langle \Phi_+(x)\Phi_+(0)\rangle &= \frac{N}{(4\pi)^2x^2}, \\
\langle \Phi_-(x)\Phi_-(0)\rangle &= \frac{2N}{(4\pi)^2x^4}, \\
\langle \Psi(x)\bar{\Psi}(0)\rangle &= \frac{N(x.\sigma)}{(4\pi)^2x^4} \\
J_B^\mu(x)J_B(0)^\nu &= \frac{N}{8\pi^2}\frac{\delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2}}{x^4} \\
J_F^\mu(x)J_F(0)^\nu &= \frac{N}{8\pi^2}\frac{\delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2}}{x^4}
\end{aligned} \tag{5.346}$$

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